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PRACTICE PROBLEMS SOLUTIONS

1. QUESTION 1 (MEDIUM). 15 POINTS

We must argue by backward induction. Consider Ann first. Let α be her choice at time 0. If $\tilde{p} = 0.4$, then $\tilde{r}_1 : (1.15, \frac{1}{2}; 0.9, \frac{1}{2})$ and $\tilde{r}_2 : (1.15, 0.4; 0.9, 0.6)$; thus, Ann will choose asset 1 [this is by first-order stochastic dominance, or because expected utility is linear in the probabilities; some argument must be provided!]. If instead $\tilde{p} = 0.54$, then she will choose asset 2. Hence, given α, Ann maximizes

$$\frac{4}{5}\left(\frac{1}{2}\sqrt{W[(1-\alpha)1.02+\alpha 1.15]}+\frac{1}{2}\sqrt{W[(1-\alpha)1.02+\alpha 0.9]}\right)+$$
$$+\frac{1}{5}\left(0.54\sqrt{W[(1-\alpha)1.02+\alpha 1.15]}+0.46\sqrt{W[(1-\alpha)1.02+\alpha 0.9]}\right).$$

It is clear that W does not affect the decision, so we can cancel it. Collecting utilities, the objective function becomes

$$\left(\frac{4}{5}\frac{1}{2}+\frac{1}{5}0.54\right)\sqrt{(1-\alpha)1.02+\alpha 1.15}+\left(\frac{4}{5}\frac{1}{2}+\frac{1}{5}0.46\right)\sqrt{(1-\alpha)1.02+\alpha 0.9}=$$
$$=0.508\sqrt{(1-\alpha)1.02+\alpha 1.15}+0.492\sqrt{(1-\alpha)1.02+\alpha 0.9}.$$

Differentiating w.r.to α, setting the result to zero and rearranging yields

$$0.254\frac{0.13}{\sqrt{(1-\alpha)1.02+\alpha 1.15}}=0.246\frac{0.12}{\sqrt{(1-\alpha)1.02+\alpha 0.9}}.$$

Further rearrangement yields

$$\sqrt{\frac{(1-\alpha)1.02+\alpha 0.9}{(1-\alpha)1.02+\alpha 1.15}}=0.894003634 \quad\Leftrightarrow\quad 1.02-0.12\alpha=0.799242498[1.02+0.13\alpha]$$

and finally

$$\alpha=\frac{1.02(1-0.799242498)}{0.12+0.799242498\cdot 0.13}=0.914565689.$$

Now consider Bob's problem. He will "reduce" probabilities and conclude that asset 2 has a return of 1.10 with probability $\mathrm{E}[\tilde{p}] = \frac{4}{5}\cdot 0.4+\frac{1}{5}\cdot 0.54 = 0.428 < \frac{1}{2}$, and therefore he will decide to invest in Asset 1 at time 1. At time 0, Bob's problem is thus pretty standard: he maximizes

$$\frac{1}{2}\sqrt{W[(1-\beta)1.02+\beta 1.15]}+\frac{1}{2}\sqrt{W[(1-\beta)1.02+\beta 0.9]}.$$

Dividing by $\frac{1}{2}\sqrt{W}$ and differentiating yields

$$\frac{1}{2}\frac{0.13}{\sqrt{(1-\beta)1.02+\beta 1.15}}-\frac{1}{2}\frac{0.12}{\sqrt{(1-\beta)1.02+\beta 0.9}}=0$$

Rearranging and squaring yields

$$\frac{(1-\beta)1.02+\beta 0.9}{(1-\beta)1.02+\beta 1.15} = 0.852071006 \Leftrightarrow 1.02-0.12\beta = 0.869112426+0.110769231*\beta$$
$$\Leftrightarrow \beta = \frac{1.02-0.869112426}{0.12+0.110769231} = 0.653846153.$$

2. QUESTION 2 (MEDIUM). 10 POINTS FOR (A), 5 POINTS FOR (B).

(a) It's easy to do this by induction—in fact, "backward induction," in a way, although there is no decision to be made except at time 0. Conditional on $X_1 = x_1, \ldots, X_{n-1} = x_{n-1}$, we must have

$$\begin{aligned}
&\mathrm{E}[u(W_0+\sum_{i=1}^n X_i)|X_1 = x_1, \ldots, X_{n-1} = x_{n-1}] = \\
=&\mathrm{E}[u(W_0+x_1+\ldots+x_{n-1}+X_n)|X_1 = x_1, \ldots, X_{n-1} = x_{n-1}] = \\
=&\mathrm{E}[u(W_0+x_1+\ldots+x_{n-1}+X_n)] < u(W_0+x_1+\ldots+x_{n-1}):
\end{aligned}$$

the first equality holds because we are conditioning on the values of the first $n-1$ repetitions, the second equality follows from the fact that the X_i's are i.i.d., and the inequality follows from the assumption that X_1 and X_n have the same distribution and $\mathrm{E}[u(W+X_1)] < u(W)$ for all $W \in [W_0 - nL, W_0 + nG]$.

Now, by the law of total probability,

$$\begin{aligned}
&\mathrm{E}[u(W_0+X_1+\ldots+X_n)] = \\
=&\sum_{x_i\in\{G,L\},i=1,\ldots,n-1} \mathrm{E}[u(W_0+\sum_{i=1}^n X_i)|X_1 = x_1, \ldots, X_{n-1} = x_{n-1}]\Pr[X_1 = x_1, \ldots, X_{n-1} = x_{n-1}] < \\
<&\sum_{x_i\in\{G,L\},i=1,\ldots,n-1} u(W_0+x_1+\ldots+x_{n-1})]\Pr[X_1 = x_1, \ldots, X_{n-1}] = \\
=&\mathrm{E}[u(W_0+X_1+\ldots+X_{n-1})].
\end{aligned}$$

In other words, you'd rather take one fewer repetition of the bet. It is clear that repeating this argument yields the result. Formally, suppose that you have shown that $\mathrm{E}[u(W_0+X_1+\ldots+X_m)] < \mathrm{E}[u(W_0+X_1+\ldots+X_{m-1})]$ for some $m \in 1, \ldots, n$ [the above argument proves it for $m = n$]. If $m = 1$, we are done [in this case, $W_0+X_1+\ldots+X_{m-1}$ means W_0, according to standard conventions]. Otherwise, note that the above arguments applies verbatim with "m" in lieu of "n": in particular, the condition $\mathrm{E}[u(W+X_1)] < u(W)$ holds for any $W \in [W_0-mL, W_0+mG] \subset [W_0-nL, W_0+nG]$, because $m \le n$. Hence, we conclude that $\mathrm{E}[u(W_0+X_1+\ldots+X_{m-1})] < \mathrm{E}[u(W_0+X_1+\ldots+X_{m-2})]$: if the above inequality holds for m, it also holds for $m-1$. This completes the proof of the inductive step.

(b) We have $\mathrm{E}[u(W_0+X_1)] = \frac{1}{2}\left(200+\frac{5}{12}200\right)+\frac{1}{2}100 = 191.666\ldots < 200 = u(W_0)$, but $\mathrm{E}[u(W_0+X_1+X_2)] = \frac{1}{4}\left(200+\frac{5}{12}400\right)+\frac{1}{2}\left(200+\frac{5}{12}100\right)+\frac{1}{4}\cdot 0 = 212.5 > 200 = u(W_0)$.

The reason this is consistent with (a) is that, if wealth is at least $W_0+200 = 400$, which includes a whole interval within the range $[W_0 - 2\cdot 100, W_0 + 2\cdot 200] = [0, 600]$, then the individual will evaluate the single instance of the bet using linear preferences, and in this case she will obviously accept it. So, the conditions in (a) are violated.

3. QUESTION 3 (EASY). 10 POINTS.

Suppose that $\text{Var}_\alpha[X] \leq \text{Var}_\alpha[Y]$ for all $\alpha \in [0,1]$. We show that Y FOSD X. Pick $x \in \mathbb{R}$: then by assumption $\text{Var}_{F_X(x)}[X] \leq \text{Var}_{F_X(x)}[Y]$, i.e. $x = F_X^{-1}(F_X(x)) \leq F_Y^{-1}(F_X(x))$, and therefore $F_Y(x) \leq F_X(x)$, i.e. $1 - F_Y(x) \geq 1 - F_X(x)$; it follows that, for all W, $W - X$ FOSD $W - Y$, and the claim follows from the characterization of FOSD we proved in class.

In the other direction, the utility characterization implies that $W - X$ FOSD $W - Y$, so Y FOSD X. Choose α: then $F_X(\text{Var}_\alpha[X]) \geq F_Y(\text{Var}_\alpha[X])$, i.e. by definition $\alpha \geq F_Y(F_X^{-1}(\alpha))$, i.e. $\text{VaR}_\alpha[Y] = F_Y^{-1}(\alpha) \geq F_X^{-1}(\alpha) = \text{VaR}_\alpha[X]$, as required.

4. QUESTION 4

(a) We must have (multiplying by -1)

$$e^{-\lambda W} = \frac{1}{2}e^{-\lambda(W-p+50)} + \frac{1}{2}e^{-\lambda(W-p)}.$$

Multiply both sides by $2e^{\lambda W}$ to get

$$2 = e^{\lambda p - \lambda 50} + e^{\lambda p}$$

Therefore

$$e^{\lambda p} = \frac{2}{e^{-\lambda 50} + 1}$$

i.e. plugging in, $e^{0.001p} = \frac{2}{e^{-0.05}+1} = 1.02499479$, so $0.001p = 0.0246875297$ and therefore $p \approx 24.6875$.

(b) Now we need

$$W^{1-\gamma} = \frac{1}{2}(W - p + 50)^{1-\gamma} + \frac{1}{2}(W-p)^{1-\gamma}.$$

Plugging in $\gamma = 2$ and $p = 15$, we get

$$W^{-1} = \frac{1}{2}(W+35)^{-1} + \frac{1}{2}(W-15)^{-1}.$$

Multiplying by $2W$ yields

$$2 = \frac{W}{W+35} + \frac{W}{W-15} = \frac{W^2 - 15W + W^2 + 35W}{W^2 + 20W - 525}$$

hence

$$2W^2 + 40W - 1050 = 2W^2 + 20W \quad \Leftrightarrow \quad W = 52.5.$$

5. QUESTION 5

(a) Denote by π_t the probability that the store assigns to Scenario 1 at the beginning of time t, as a function of its observations to date. Thus, $\pi_1 = 0.6$ and π_2 is a random variable whose realization depends upon the store's choice of product and whether or not there was a sale.

We use backward induction. At time 2, the store must choose myopically: it will offer A if $\pi_2 \geq \frac{1}{2}$ and B otherwise. Therefore, if $\pi_2 \geq \frac{1}{2}$, the store's expected payoff will be

$$\pi_t[0.75 \cdot 1 + 0.25 \cdot 0] + (1 - \pi_t)[0.75 \cdot 0 + 0.25 \cdot 1] = \pi_t 0.75 + (1 - \pi_t)0.25 = 0.25 + 0.5\pi_t;$$

if instead $\pi_2 < \frac{1}{2}$, then the expected payoff is

$$(1 - \pi_t)(0.75 \cdot 1 + 0.25 \cdot 0) + \pi_t(0.75 \cdot 0 + 0.25 \cdot 1) = (1 - \pi_t)0.75 + \pi_t 0.25 = 0.75 - 0.5\pi_t.$$

Now consider time 1. If the store chooses A and a sale occurs, then the store will have a payoff of 1 at time 1, and update her beliefs about the likelihood of Scenario 1 using Bayes' rule:

$$\pi_2 = \frac{0.75 \cdot 0.6}{0.75 \cdot 0.6 + 0.25 \cdot 0.4} = .81818181818181818181,$$

so the store will continue to choose A at time 2. If instead a sale does not occur, then there is no payoff at time 0, and furthermore

$$\pi_2 = \frac{0.25 \cdot 0.6}{0.25 \cdot 0.6 + 0.75 \cdot 0.4} = .33333333333333333333$$

so that the store will choose B at time 2. Finally, the ex-ante probability of a sale at time 1 if the store chooses A is

$$0.75 \cdot 0.6 + 0.25 \cdot 0.4 = .55$$

and therefore the total expected payoff from choosing A at time 1 is (plugging in the formulas for payoff at time 2)

$$0.55 \cdot (1 + 0.25 + 0.5 \cdot .8181818181818181) + 0.45 \cdot (0 + 0.75 - 0.5 \cdot .3333333333333333) = 1.175.$$

If instead the store chooses B, and a sale occurs, then

$$\pi_2 = \frac{0.25 \cdot 0.6}{0.25 \cdot 0.6 + 0.75 \cdot 0.4} = .33333333333333333333$$

so the store will continue with B; and if a sale does not occur at time 1, then

$$\pi_2 = \frac{0.75 \cdot 0.6}{0.75 \cdot 0.6 + 0.25 \cdot 0.4} = .8181818181818181$$

so A will be optimal in the continuation (note the symmetry). The probability of a sale is

$$0.25 \cdot 0.6 + 0.75 \cdot 0.4 = .45$$

Hence, the total expected payoff from B at time 1 is

$$.45 \cdot (1 + 0.75 - 0.5 \cdot 0.3333333333333333) + 0.55 \cdot (0 + 0.25 + 0.5 \cdot 0.8181818181818181) = 1.075.$$

It is therefore optimal to choose A at time 1.

(b) Yes, it turns out that the myopic policy is optimal in this case.

6. QUESTION 6

(a) Let $X : (150, 0.4; 80, 0.6)$. Without any newsletter, the investor compares

$$-0.4e^{-\lambda(W+(150-100)50)} - 0.6e^{-\lambda(W+(80-100)50)}.$$

with $-e^{-\lambda W}$. Dividing by $e^{-\lambda W}$ and plugging in $\lambda = 0.001$, she compares

$$-0.4e^{-0.001 \cdot 2500} - 0.6e^{0.001 \cdot 1000} = -0.4e^{-2.5} - 0.6e^{1} = -1.663803$$

with -1. Thus, she should *not* buy the shares.

(b) Let's consider BadNews first. We need to compute the probability of H in case of a bad report:

$$\Pr[X = H|B = L] = \frac{\Pr[B = L|X = H]\Pr[X = H]}{\Pr[B = L|X = H]\Pr[X = H] + \Pr[B = L|X = L]\Pr[X = L]} = \frac{0.4 \cdot 0.4}{0.4 \cdot 0.4 + 0.9 \cdot 0.6} = 0.2285714.$$

Hence, in case of a bad report, we are comparing

$$-0.2285714e^{-2.4} - (1 - 0.2285714)e^{1.1} = -2.33823512$$

with $-e^{-\lambda\cdot(-100)} = -e^{0.1} = -1.10517092$, because at this stage we have already paid for the report; thus, we would not buy; in case of a good report,

$$\Pr[X = H|B = H] = \frac{\Pr[B = H|X = H]\Pr[X = H]}{\Pr[B = H|X = H]\Pr[X = H] + \Pr[B = H|X = L]\Pr[X = L]} = \frac{0.6 \cdot 0.4}{0.6 \cdot 0.4 + 0.1 \cdot 0.6} = 0.8,$$

and we compare -1.10517092 with

$$-0.8e^{-2.4} - 0.2e^{1.1} = -0.673407567$$

so in this case we *do* buy. In other words, if the investor gets the BadNews signal, she does not buy in case of bad news ($B = L$), and does buy in case of good news ($B = H$). Now the ex-ante probability of good news is

$$\Pr[B = H] = \Pr[B = H|X = H]\Pr[X = H]+\Pr[B = H|X = L]\Pr[X = L] = 0.6\cdot 0.4+0.1\cdot 0.6 = 0.3,$$

so the ex-ante expected payoff in case the investor buys the BadNews signal is

$$0.3 \cdot (-0.673407567) + 0.7 \cdot (-1.10517092) = -0.975641914$$

which is better than not buying the signal and taking the optimal decision in (a), namely not buying the shares (recall this nets utility -1).

Now let's consider GoodNews. In case of a H signal,

$$\Pr[X = H|G = H] = \frac{0.9 \cdot 0.4}{0.9 \cdot 0.4 + 0.4 \cdot 0.6} = 0.6,$$

and we compare

$$-0.6e^{-2.45} - 0.4e^{1.05} = -1.1948366$$

with $-e^{0.05} = -1.0512711$: thus, with a H signal, the investor does not buy. A fortiori, she will not buy in case of a L signal, as in that case

$$\Pr[X = H|G = L] = \frac{0.1 \cdot 0.4}{0.1 \cdot 0.4 + 0.6 \cdot 0.6} = 0.1$$

i.e. the probability of a good outcome is lower than if $G = H$. Hence, the investor will not buy no matter what the realization of G, which means that G is worthless. So, she will *not* buy GoodNews.