

Math 1132 Midterm-1 Review.

①

5.5 substitutions.

$$1 \text{ a. } \int \frac{x}{\sqrt{x-4}} dx \quad \int \frac{x}{\sqrt{x-4}} dx = \int \frac{(u+4) du}{\sqrt{u}}$$

$$u = x - 4$$

$$\frac{du}{dx} = 1 \cdot du = dx$$

$$x = u + 4$$

$$\begin{aligned} \int \left(\frac{u}{\sqrt{u}} + \frac{4}{\sqrt{u}} \right) du &= \int u^{1/2} du + \int 4 \cdot u^{-1/2} du \\ &= \frac{u^{3/2}}{3/2} + 4 \cdot \frac{u^{1/2}}{1/2} + C \end{aligned}$$

$$= \frac{2}{3} u^{3/2} + 8 \cdot u^{1/2} + C$$

$$= \frac{2}{3} (x-4)^{3/2} + 8(x-4)^{1/2} + C$$

$$2. b \quad \int_0^2 \frac{2x}{(x^2+1)^2} dx \quad \begin{aligned} u &= x^2 + 1 & x = 0 & u = 1 \\ du &= 2x dx & x = 2 & u = 5 \end{aligned}$$

$$\int_1^5 \frac{du}{u^2} = \int_1^5 u^{-2} du = -\frac{1}{u} \Big|_1^5 = -\frac{1}{5} + 1 = \frac{4}{5}.$$

(3)

$$1.C \quad \int \cos \theta \sin^6 \theta d\theta$$

$$= \int \sin^6 \theta \cdot \cos \theta d\theta$$

$$u = \sin \theta \quad du = \cos \theta d\theta$$

$$\int u^6 du = \frac{u^7}{7} + C = \frac{\sin^7 \theta}{7} + C$$

$$2.D \quad \int \sqrt{\cot x} \cdot \csc^2 x dx$$

$$u = \cot x \quad du = -\csc^2 x dx$$

$$\int \sqrt{u} (-du) = - \int u^{1/2} du = - \frac{u^{3/2}}{3/2} + C$$

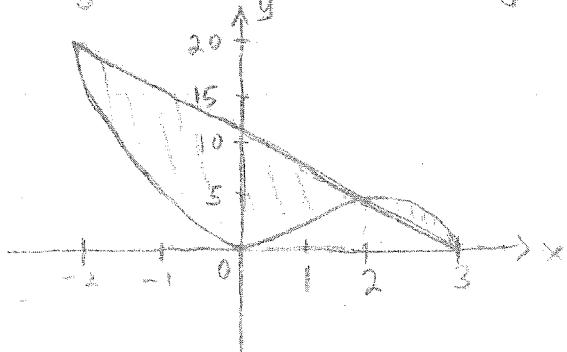
$$= -\frac{2}{3} u^{3/2} + C = -\frac{2}{3} (\cot x)^{3/2} + C$$

$$1.E \quad \int \frac{\csc^2 x}{1 + \cot x} dx \quad \begin{aligned} u &= 1 + \cot x \\ du &= -\csc^2 x dx \end{aligned}$$

$$\begin{aligned} \int -\frac{du}{u} &= -\ln|u| + C \\ &= -\ln|1 + \cot x| + C \end{aligned}$$

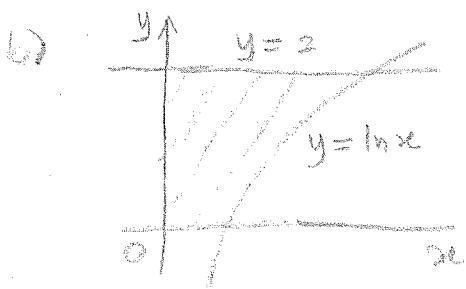
2. 6.2: Areas under the curve solutions

2 (a) $y = x^2(3-x)$ and $y = 12 - 4x$

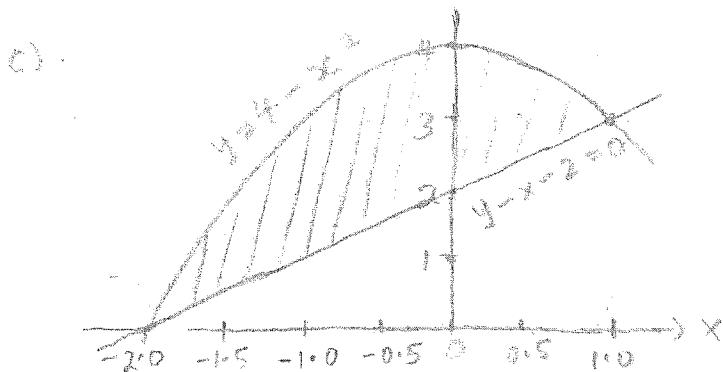


$$A = \int_{-2}^2 (12 - 4x - (3x^2 - x^3)) dx + \int_{-2}^3 (3x^2 - x^3 - (12 - 4x)) dx$$

$$\begin{aligned} &= \left[12x - 2x^2 - x^3 + \frac{x^4}{4} \right]_{-2}^2 + \left(x^3 - \frac{x^4}{4} - 12x + 2x^2 \right) \Big|_2 \\ &= 24 - 8 - 8 + 4 - (-44 - 8 + 8 + 4) + (27 - 8\frac{1}{4} - 36 + 16 - (2 - 4 + 24 + 8)) = \boxed{\frac{31}{4}} \end{aligned}$$



$$A = \int_0^2 e^y dy = e^y \Big|_0^2 = \boxed{e^2 - 1}$$

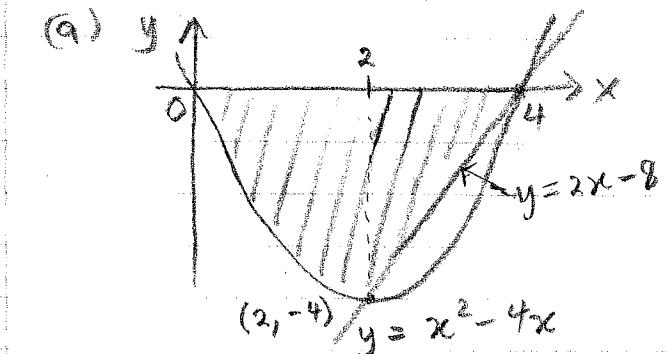


$$\begin{aligned} A &= \int_{-2}^2 [(4-x^2) - (x+2)] dx \\ &= \int_{-2}^2 (2-x^2-x) dx \end{aligned}$$

$$\begin{aligned} &= \left[2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{-2}^1 = \left(2 - \frac{1}{3} - \frac{1}{2} \right) - \left(-4 + 8 - 2 \right) \\ &= \boxed{\frac{9}{2}} \end{aligned}$$

3. $y = x^2 - 4x$, $y = 2x - 8$, and the x -axis.

(a) $y \uparrow$



pt. of intersection:

$$x^2 - 4x = 2x - 8$$

$$x^2 - 6x + 8 = 0$$

$$(x-4)(x-2) = 0$$

$$x = 2, 4.$$

$$\text{So } A = \int_0^2 (0 - (x^2 - 4)) dx + \int_2^4 (0 - (2x - 8)) dx$$

$$= \int_0^2 (-x^2 + 4) dx + \int_2^4 (-2x + 8) dx = \underline{x^3}$$

$$= \left[-\frac{x^3}{3} + 4x \right]_0^2 + \left[-x^2 + 8x \right]_2^4$$

$$= -\frac{8}{3} + 8 - 0 + (-16 + 32 - (-4 + 16)) = \boxed{\frac{28}{3}}$$

(b) $A = \int_{y=-4}^{y=0} (x_L - x_R) dy$ Note $x_L - x$ on the left
 $x_R - x$ on the right

$$\text{Right equation: } y = 2x - 8 \Rightarrow x_R = (y + 8)/2 = \frac{y}{2} + 4$$

Left equation: $y = x^2 - 4x$, we solve for x by completing the square to obtain

$$y + 4 = x^2 - 4x + 4 = (x-2)^2$$

$$\Rightarrow x = 2 \pm \sqrt{y+4}$$

The part of the curve we need must satisfy only the points from $(0,0)$ to $(2, -4)$, since that is the left equation, and that is $x_L = 2 - \sqrt{y+4}$. (example: when $y=0$)

$$\begin{aligned}
 \text{so } A &= \int_{-4}^0 \left(\frac{y}{2} + 4 \right) - \left(2 - \sqrt{y+4} \right) dy \\
 &= \int_{-4}^0 \left(\frac{y}{2} + 2 + \sqrt{y+4} \right) dy \\
 &= \left[\frac{y^2}{4} + 2y + \frac{2}{3}(y+4)^{3/2} \right]_{-4}^0 \\
 &= 0 + 0 + \frac{2}{3}(4)^{3/2} - (4 - 8 + 0) \\
 &= \frac{16}{3} + 4 = \boxed{\frac{28}{3}}
 \end{aligned}$$

1. Let S be the region bounded by $y = x^3$, $y = 1$, and $x \geq 0$. What is the volume of the solid obtained by rotating S about the line $y = 1$?

Solution: The radius is given by $1 - x^3$ and the bounds are $x = 0$ and $x = 1$. Hence the volume is

$$\begin{aligned}\pi \int_0^1 (1 - x^3)^2 dx &= \pi \int_0^1 1 - 2x^3 + x^6 dx \\ &= \frac{9\pi}{14}.\end{aligned}$$

2. Let S be the region bounded by the curves $y = x^2$, $y = x$ and $x \geq 0$. What is the volume of the solid obtained by rotating S about

- (a) the x-axis?

Solution: Using the washer method, the outer radius is given by $r_o = x$, the inner radius by $r_i = x^2$, and the bounds are $x = 0$ and $x = 1$. Hence the volume is

$$\begin{aligned}\pi \int_0^1 x^2 - (x^2)^2 dx &= \pi \int_0^1 x^2 - x^4 dx \\ &= \frac{2\pi}{15}.\end{aligned}$$

- (b) the y-axis?

Solution: Using the washer method, the outer radius is given by $r_o = \sqrt{y}$, the inner radius by $r_i = y$, and the bounds are $y = 0$ and $y = 1$. Hence the volume is

$$\begin{aligned}\pi \int_0^1 (\sqrt{y})^2 - y^2 dy &= \pi \int_0^1 y - y^2 dy \\ &= \frac{\pi}{6}.\end{aligned}$$

3. Let S be the region bounded by $y = x$, $y = 2x$ and $y = 2$. What is the volume of the solid obtained by rotating S about the y-axis?

Solution: Using the washer method, the outer radius is given by $r_o = y$, the inner radius by $r_i = \frac{y}{2}$, and the bounds are $y = 0$ and $y = 2$. Hence the volume is

$$\begin{aligned}\pi \int_0^2 y^2 - \left(\frac{y}{2}\right)^2 dy &= \frac{3\pi}{4} \int_0^2 y^2 dy \\ &= 2\pi.\end{aligned}$$

4. Consider the region S bounded by the parabolas $y = x^2 - 1$ and $y = 1 - x^2$. Find the volume of the solid with base S and cross-sections perpendicular to S that are squares with bases parallel to the y-axis.

Solution: The curves intersect when $x = -1$ and $x = 1$. Taking advantage of symmetry, the base length of any cross-section is $2(1 - x^2)$. Hence the area of any cross-section is

$$[2(1 - x^2)]^2 = 4(1 - 2x^2 + x^4).$$

The volume integral is then

$$\begin{aligned}4 \int_{-1}^1 1 - 2x^2 + x^4 dx &= 8 \int_0^1 1 - 2x^2 + x^4 dx \quad (\text{note symmetry}) \\ &= \frac{64}{15}.\end{aligned}$$

5. Let S be given as in the previous problem. Find the volume of the solid with base S and cross-sections perpendicular to S that are equilateral triangles with bases parallel to the y -axis.

Solution: From (4), the base length of a cross-section is $2(1 - x^2)$. From triangle geometry, the height of a cross-section is $\sqrt{3}(1 - x^2)$. Hence the area of any cross-section is

$$\begin{aligned}\frac{1}{2}bh &= \frac{1}{2} [2(1 - x^2)\sqrt{3}(1 - x^2)] \\ &= \sqrt{3}(1 - x^2)^2 \\ &= \sqrt{3}(1 - 2x^2 + x^4).\end{aligned}$$

The volume integral is then

$$\begin{aligned}\sqrt{3} \int_{-1}^1 1 - 2x^2 + x^4 dx &= 2\sqrt{3} \int_0^1 1 - 2x^2 + x^4 dx \quad (\text{note symmetry}) \\ &= \frac{16\sqrt{3}}{15}.\end{aligned}$$

6. Using calculus, find the length of $y = 6x + 2$ on the interval $0 \leq x \leq 1$.

Solution: Letting $y = f(x)$, we have $f'(x) = y' = 6$ so that the arc length is

$$\begin{aligned}\int_0^1 \sqrt{1 + f'(x)^2} dx &= \int_0^1 \sqrt{1 + 6^2} dx \\ &= \sqrt{37} \int_0^1 dx \\ &= \sqrt{37}.\end{aligned}$$

7. Let $y = \frac{x^4}{4} + \frac{x^{-2}}{8}$ and find the length of y on the interval $1 \leq x \leq 2$.

Solution: Letting $y = f(x)$, we have $f'(x) = y' = x^3 - \frac{x^{-3}}{4}$ so that

$$f'(x)^2 = x^6 - \frac{1}{2} + \frac{x^{-6}}{16}$$

so that

$$\begin{aligned}1 + f'(x)^2 &= x^6 + \frac{1}{2} + \frac{x^{-6}}{16} \\ &= \left(x^3 + \frac{x^{-3}}{4}\right)^2.\end{aligned}$$

Therefore, the arc length is

$$\begin{aligned}\int_1^2 \sqrt{1 + f'(x)^2} dx &= \int_1^2 x^3 + \frac{x^{-3}}{4} dx \\ &= \frac{123}{32}\end{aligned}$$

8. Let $y = \frac{x^2}{4} - \frac{\ln x}{2}$ and find the length of y on the interval $1 \leq x \leq e^2$.

Solution: Letting $y = f(x)$, we have $f'(x) = y' = \frac{x}{2} - \frac{1}{2x}$ so that

$$f'(x)^2 = \frac{x^2}{4} - \frac{1}{2} + \frac{x^{-2}}{4}$$

so that

$$\begin{aligned}1 + f'(x)^2 &= \frac{x^2}{4} + \frac{1}{2} + \frac{x^{-2}}{4} \\ &= \left(\frac{x}{2} + \frac{x^{-1}}{2}\right)^2.\end{aligned}$$

Therefore, the arc length is

$$\begin{aligned}\int_1^{e^2} \sqrt{1 + f'(x)^2} dx &= \int_1^{e^2} \frac{x}{2} + \frac{x^{-1}}{2} dx \\ &= \frac{e^4 + 3}{4}\end{aligned}$$

9. Find the length of the curve $y = 4x^{\frac{3}{2}}$ on the interval $0 \leq x \leq 2$.

Solution: Letting $y = f(x)$, we have $f'(x) = y' = 2\sqrt{x}$ so that

$$1 + f'(x)^2 = 1 + 4x.$$

Therefore, the arc length is

$$\int_0^2 \sqrt{1 + 4x} dx.$$

Letting $u = 1 + 4x$, we have $\frac{1}{4}du = dx$, $u(0) = 1$, and $u(2) = 9$ so that

$$\begin{aligned}\int_0^2 \sqrt{1 + 4x} dx &= \frac{1}{4} \int_1^9 u^{\frac{1}{2}} du \\ &= \frac{13}{3}.\end{aligned}$$

10. Find the length of the curve $y = x^{\frac{2}{3}}$ for $x \geq 0$ on the interval $0 \leq y \leq 4$.

Solution: Notice that the bounds are in terms of y , so we are going to integrate with respect to y . If we rewrite the formula as $x = y^{\frac{3}{2}}$ and let $g(y) = x$, then $g'(y) = x' = \frac{3\sqrt{y}}{2}$ so that

$$1 + g'(y)^2 = 1 + \frac{9}{4}y.$$

Therefore, the arc length is

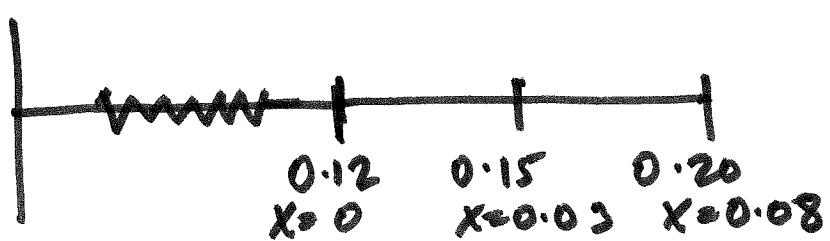
$$\int_0^4 \sqrt{1 + \frac{9}{4}y} dy.$$

Letting $u = 1 + \frac{9}{4}y$, we have $\frac{9}{4}du = dy$, $u(0) = 1$, and $u(4) = 10$ so that

$$\begin{aligned}\int_0^4 \sqrt{1 + \frac{9}{4}y} dy &= \frac{4}{9} \int_1^{10} u^{\frac{1}{2}} du \\ &= \frac{8}{27} (10^{\frac{3}{2}} - 1).\end{aligned}$$

Applications to Physics.

13. $F = 30 \text{ N.}$



$$F = kx$$

$$30 = 0.03 \cdot k$$

$$k = \frac{30}{0.03} = 1000$$

$$W = \int F dx = \int_0^{0.08} 1000 \cdot x \cdot dx = 1000 \int_0^{0.08} x dx$$

$$= 1000 \cdot \frac{x^2}{2} \Big|_0^{0.08} = \frac{1000}{2} (0.08)^2 = 3.2 \text{ J.}$$

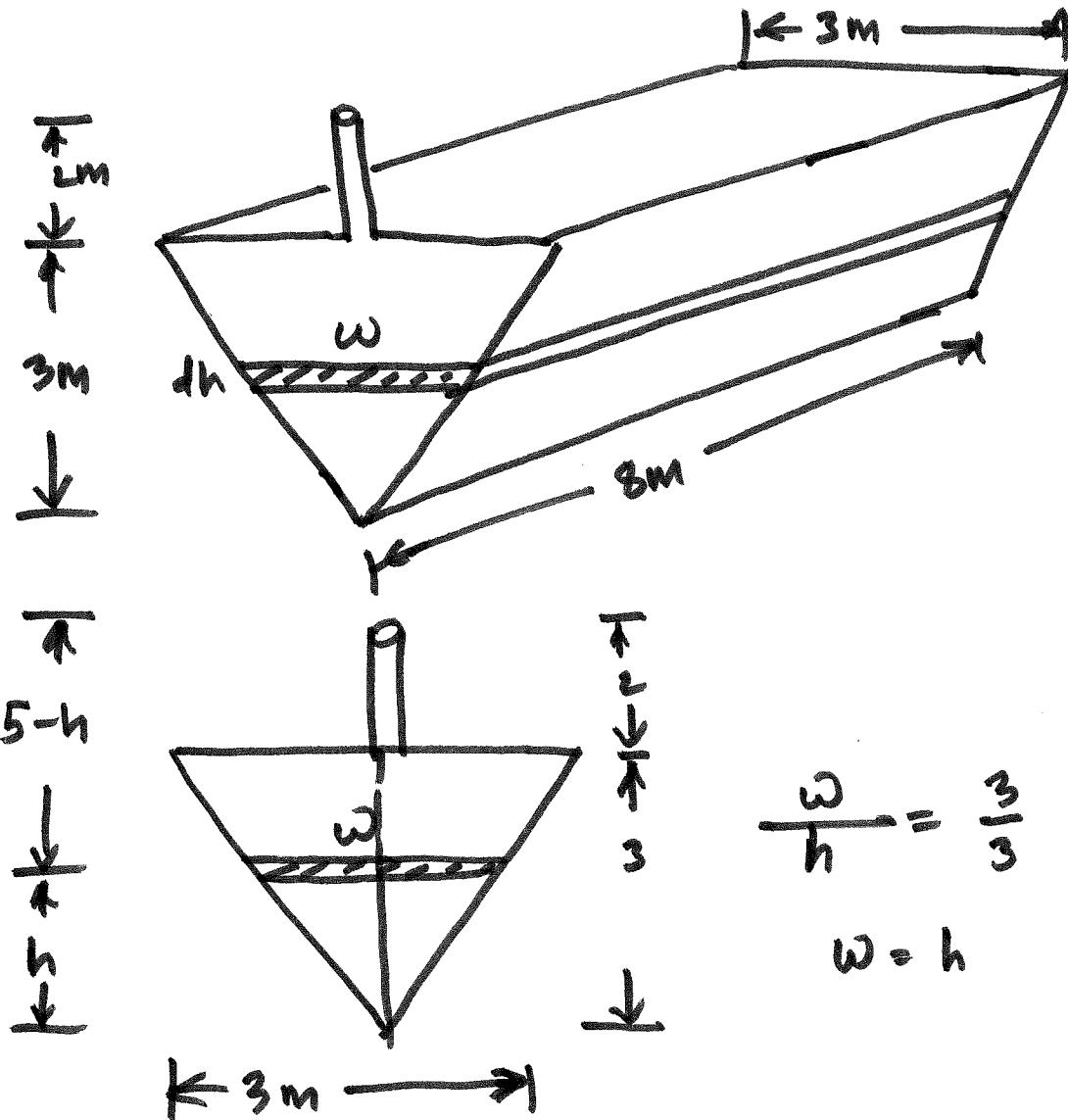
14. $W = \int F dx = \int_0^{0.8} kx \cdot dx = k \int_0^{0.8} x dx = k \frac{x^2}{2} \Big|_0^{0.8}$

$$160 = k \cdot \frac{(0.8)^2}{2} \quad k = \frac{320}{(0.8)^2} = 500.$$

$$W = \int F dx = \int_{0.8}^{1.2} 500x dx = 500 \frac{x^2}{2} \Big|_{0.8}^{1.2}$$

$$= \frac{500}{2} (1.2^2 - 0.8^2) = 200 \text{ J.}$$

15



$$\text{Area}_{\text{slice}} = w \cdot 8 = 8 \cdot h$$

$$\text{Vol}_{\text{slice}} = 8 \cdot h \cdot dh$$

$$\begin{aligned} \text{mass} &= 8 \cdot h \cdot dh \cdot \text{density} = 8 \cdot h \cdot dh \cdot 1000 \\ &= 8000 h dh \end{aligned}$$

$$F = \text{mass} \cdot \text{gravity acc}^h$$

$$= 8000 h dh \cdot 9.8$$

$$W = F \cdot \text{dist} = 8000 h dh \cdot 9.8 (5-h)$$

$$\begin{aligned}
 \text{Total work} &= \int_0^3 78400 h(5-h) dh \\
 &= 78400 \int_0^3 (5h - h^2) dh \\
 &= 78400 \left[\frac{5h^2}{2} - \frac{h^3}{3} \right]_0^3 \\
 &= 78400 \left[\frac{5 \cdot 9}{2} - \frac{27}{3} \right] = 1.0584 \times 10^6 \text{ J}
 \end{aligned}$$

(16)



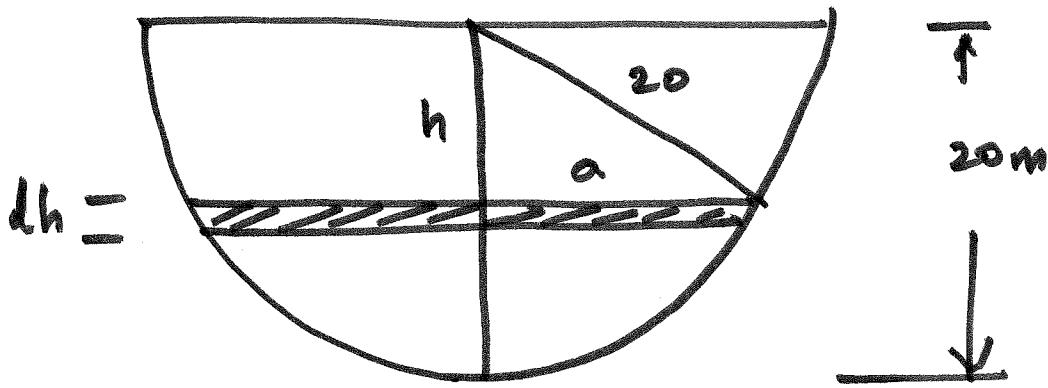
x_0

dx

$x=3$

$$\begin{aligned}
 m &= \int_0^3 \rho dx = \int_0^3 150 e^{-\frac{x}{3}} dx \\
 &= 150 \cdot \frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \Big|_0^3 = -450 [e^{-1} - e^0] \\
 &= 284.45
 \end{aligned}$$

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$$\text{Width of ship} = w = 2a$$

$$\text{Area} = dh \cdot 2a$$

$$a = \sqrt{400 - h^2}$$

$$A = 2\sqrt{400 - h^2} \cdot dh$$

$$P = \rho g A \text{ l. g. depth} = 1000 \cdot 9.8 \cdot h = 9800h$$

$$F = P \cdot A = 9800h \cdot 2\sqrt{400 - h^2} dh$$

$$F = \int_0^{20} 9800 \sqrt{400 - h^2} 2h dh \quad u = 400 - h^2 \\ du = -2h dh$$

$$F = - \left(9800 u^{3/2} \right) \Big|_0^{400} \quad h=0 \quad u=400 \\ h=20 \quad u=0$$

$$F = -9800 \left[\frac{u^{3/2}}{3/2} \right]_{400}^{0} = -\frac{9800 \cdot 2}{3} \left[0^{3/2} - 400^{3/2} \right] \\ = -\frac{9800 \cdot 2}{3} (-20)^3 = 5.22 \times 10^7 \text{ N. } \blacksquare$$

6. Integration by Parts.

18. a) $\int 3x \cdot \sec^2 x \, dx$.

$$u = 3x \quad dv = \sec^2 x \, dx$$

$$u' = 3 \, dx \quad v = \tan x.$$

$$\begin{aligned}\int 3x \cdot \sec^2 x \, dx &= 3x \cdot \tan x - \int 3 \cdot \tan x \, dx \\ &= 3x \cdot \tan x + 3 \ln |\cos x| + C\end{aligned}$$

b) $\int x^2 e^{5x} \, dx$

$$u = x^2 \quad dv = e^{5x} \, dx$$

$$du = 2x \, dx \quad v = \frac{e^{5x}}{5}$$

$$\int x^2 e^{5x} \, dx = x^2 \cdot \frac{e^{5x}}{5} - \int 2x \cdot \frac{e^{5x}}{5} \, dx$$

$$u = 2x \quad dv = \frac{e^{5x}}{5} \, dx$$

$$du = 2 \, dx \quad v = \frac{e^{5x}}{25}$$

$$\int x^2 e^{5x} \, dx = \frac{x^2 \cdot e^{5x}}{5} - \frac{2x \cdot e^{5x}}{25} + \int \frac{2e^{5x}}{25} \, dx$$

$$= \frac{x^2 e^{5x}}{5} - \frac{2x e^{5x}}{25} + \frac{2}{125} e^{5x} + C.$$

18c

$$\int e^{2x} \cos 4x \, dx$$

$$u = \cos 4x \quad dv = e^{2x} \, dx$$

$$du = -\sin 4x \cdot 4 \, dx \quad v = \frac{e^{2x}}{2}$$

$$\int e^{2x} \cos 4x \, dx = \frac{e^{2x}}{2} \cos 4x - \int \frac{e^{2x}}{2} \cdot (-\sin 4x) \cdot 4 \, dx$$

$$= \frac{e^{2x}}{2} \cos 4x + \int 2e^{2x} \sin(4x) \, dx$$

$$u = \sin 4x \quad dv = 2e^{2x} \, dx$$

$$du = \cos 4x \cdot 4 \, dx \quad v = 2 \cdot \frac{e^{2x}}{2}$$

$$\int e^{2x} \cos 4x \, dx = \frac{e^{2x}}{2} \cos 4x + e^{2x} \cdot \sin 4x - 4 \int e^{2x} \cos 4x \, dx \\ + 4 \int e^{2x} \cos 4x \, dx$$

$$+ 4 \int e^{2x} \cos 4x \, dx$$

$$5 \int e^{2x} \cos 4x \, dx = \frac{e^{2x}}{2} \cos 4x + e^{2x} \sin 4x$$

$$\int e^{2x} \cos 4x \, dx = \frac{e^{2x}}{10} \cos 4x + \frac{1}{5} e^{2x} \sin 4x + C$$

$$(18) \text{ d. } \int x^3 \ln x \, dx$$

$$u = \ln x \quad dv = x^3 \, dx$$

$$du = \frac{1}{x} \, dx \quad v = \frac{x^4}{4}$$

$$\int x^3 \ln x \, dx = \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} \cdot dx$$

$$= \frac{\ln x \cdot x^4}{4} - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{\ln x \cdot x^4}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + C$$

$$= \frac{x^4}{4} \left[\ln x - \frac{1}{4} \right] + C$$

$$\text{e)} \quad \int_0^{\pi/2} (x-4) \sin x \, dx$$

$$= \int_0^{\pi/2} x \sin x \, dx - \int_0^{\pi/2} 4 \sin x \, dx$$

$$I_1 = \int_0^{\pi/2} x \sin x \, dx \quad I_2$$

$$u = x \quad dv = \sin x \, dx$$

$$du = dx \quad v = -\cos x$$

$$\begin{aligned}
 I_1 &= \int_0^{\pi/2} x \sin x \, dx = x \cdot (-\cos x) \Big|_0^{\pi/2} - \int_0^{\pi/2} -\cos x \, dx \\
 &= -x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx \\
 &= -x \cos x \Big|_0^{\pi/2} + \sin x \Big|_0^{\pi/2} \\
 &= [-\pi/2 \cdot \cos \pi/2 + 0 \cdot \cos 0] + [\sin \pi/2 - \sin 0]
 \end{aligned}$$

0 + 1.

$$\begin{aligned}
 I_2 &= \int_0^{\pi/2} 4 \sin x \, dx = 4[-\cos x]_0^{\pi/2} \\
 &= 4[-\cos \pi/2 + \cos 0] = -4.
 \end{aligned}$$

$$\int_0^{\pi/2} (x-4) \sin x \, dx = I_1 + I_2 = 1 + 4 = 5.$$

7. 7.2 Trigonometric Integrals.

19. a) $\int_0^{\pi} \cos^6 \theta d\theta$.

$$\int_0^{\pi} (\cos^2 \theta)^3 d\theta = \int_0^{\pi} \left(\frac{1+\cos 2\theta}{2}\right)^3 d\theta$$

$$= \frac{1}{8} \int_0^{\pi} (1+\cos 2\theta)^3 d\theta$$

$$= \frac{1}{8} \int_0^{\pi} (1 + 3\cos 2\theta + 3\cos^2 2\theta + \cos^3 2\theta) d\theta$$

$$I_1 = \int_0^{\pi} 1 d\theta \quad I_2 = \int_0^{\pi} 3\cos 2\theta \rightarrow I_3 = \int_0^{\pi} 3\cos^2 2\theta$$

$$I_4 = \int_0^{\pi} \cos^3 2\theta d\theta$$

$$I_1 = \int_0^{\pi} 1 d\theta = \theta \Big|_0^{\pi} = \pi.$$

$$I_2 = \int_0^{\pi} 3\cos 2\theta = 3 \cdot \frac{\sin 2\theta}{2} \Big|_0^{\pi} = \frac{3}{2} [\sin(2\pi) - \sin 0] \\ = \frac{3}{2} \cdot 0 = 0$$

$$I_3 = \int_0^{\pi} 3 \cos^2 2\theta = \int_0^{\pi} 3 \left(\frac{1 + \cos 4\theta}{2} \right) d\theta$$

$$= \frac{3}{2} \left[\int_0^{\pi} 1 d\theta + \int_0^{\pi} \cos 4\theta d\theta \right]$$

$$= \frac{3}{2} \left[\theta \Big|_0^{\pi} + \frac{\sin 4\theta}{4} \Big|_0^{\pi} \right]$$

$$= \frac{3}{2} \left[(\pi - 0) + \frac{1}{4} (\sin 4\pi - \sin 0) \right]$$

$$= \frac{3}{2} \pi$$

$$I_4 = \int_0^{\pi} \cos^3 2\theta d\theta = \int_0^{\pi} \cos^2 2\theta \cdot \cos 2\theta d\theta$$

$$= \int_0^{\pi} (1 - \sin^2 2\theta) \cos 2\theta d\theta \quad u = \sin 2\theta \\ du = \cos 2\theta \cdot 2 d\theta$$

$$u=0 \quad \theta=0 \\ u=0 \quad \theta=\pi.$$

$$\therefore I_4 = 0.$$

$$I = \frac{1}{8} [I_1 + I_2 + I_3 + I_4]$$

$$= \frac{1}{8} \left[\pi + 0 + \frac{3}{2} \pi + 0 \right] = \frac{1}{8} \cdot \frac{5\pi}{2} = \frac{5\pi}{16}$$

$$19 \quad b) \int_{\pi/2}^{3\pi/4} \sin^5 x \cdot \cos^3 x \, dx$$

$$\int_{\pi/2}^{3\pi/4} \sin^5 x \cdot (1 - \sin^2 x) \cdot \cos x \, dx$$

$$u = \sin x \quad du = \cos x \, dx$$

$$\text{when } x = \pi/2 \quad u = \sin(\pi/2) = 1$$

$$x = 3\pi/4 \quad u = \sin(3\pi/4) = \sqrt{2}$$

$$\int_1^{\sqrt{2}} u^5 (1 - u^2) \, du = \int_1^{\sqrt{2}} (u^5 - u^7) \, du$$

$$\begin{aligned} \left. \frac{u^6}{6} - \frac{u^8}{8} \right|_1^{\sqrt{2}} &= \frac{1}{6} \left[\frac{1}{8} - 1 \right] - \frac{1}{8} \left[\frac{1}{16} - 1 \right] \\ &= \frac{1}{6} \left[-\frac{7}{8} \right] - \frac{1}{8} \left[-\frac{15}{16} \right] \\ &= -\frac{7}{48} + \frac{15}{8 \cdot 16} = \frac{-11}{384}. \end{aligned}$$

$$19 \quad c) \int \tan^5 x \sec^4 x \, dx$$

$$\int \tan^5 x \cdot \sec^2 x \cdot \sec^2 x \, dx$$

$$\int \tan^5 x \cdot (1 + \tan^2 x) \cdot \sec^2 x \, dx$$

$$u = \tan x \quad du = \sec^2 x \, dx$$

$$\begin{aligned} \int u^5(1+u^2)du &= \int(u^5 + u^7)du \\ &= \frac{u^6}{6} + \frac{u^8}{8} + C \\ &= \frac{\tan^6 x}{6} \neq \frac{\tan^8 x}{8} + C \end{aligned}$$

19) $\int \cos^3(9x) \sin^{-2}(9x) dx$

$$\begin{aligned} \int \frac{\cos^3 9x}{\sin^2 9x} dx &= \int \frac{\cos^2 9x \cdot \cos 9x}{\sin^2 9x} dx \\ &= \int \frac{(1 - \sin^2 9x)}{\sin^2 9x} \cdot \cos 9x dx \quad u = \sin 9x \\ &\quad du = \cos 9x \cdot 9 dx \\ \int \frac{(1-u^2)}{u^2} \cdot \frac{du}{9} &= \frac{1}{9} \int \left(\frac{1}{u^2} - 1\right) du \\ &= \frac{1}{9} \left[\int u^{-2} du - \int du \right] \\ &= \frac{1}{9} \left[-u^{-1} - u \right] \\ &= \frac{1}{9} \left[-\frac{1}{u} - u \right] + C \\ &= \frac{1}{9} \left[-\frac{1}{\cos(9x)} - \frac{1}{9} \right] + C \end{aligned}$$

19 E.

$$\int \cot^3 x \cdot \csc^3 x \, dx$$

$$\int \cot^2 x \cdot \csc^2 x \cdot \cot x \csc x \, dx$$

$$\int (\csc^2 x - 1) \cdot \csc^2 x \cot x \csc x \, dx$$

$$\int u = -\cot x \csc x \, dx$$

$u = \csc x \quad du = -\cot x \csc x \, dx$

$$\int (u^2 - 1)(-du) = \int (1 - u^2) du$$

$$= \cancel{u} + u - \frac{u^3}{3} + C$$

$$= \csc x - \frac{\csc^3 x}{3} + C$$

8. 7.3 Trigonometric Substitution

$$20 \int \frac{1}{x^2 \sqrt{25-x^2}} dx$$

$$x = 5 \sin \theta$$

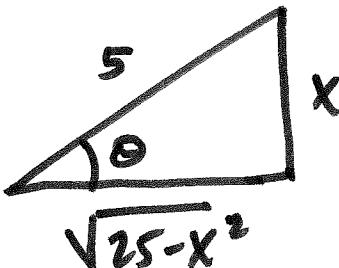
$$1. \quad dx = 5 \cos \theta d\theta$$

$$\begin{aligned} 2. \quad \frac{1}{x^2 \sqrt{25-x^2}} &= \frac{1}{25 \sin^2 \theta \sqrt{25-25 \sin^2 \theta}} \\ &= \frac{1}{25 \sin^2 \theta \sqrt{25 \cos^2 \theta}} = \frac{1}{25 \cdot \sin^2 \theta \cdot 5 \cdot \cos \theta} \end{aligned}$$

$$\int \frac{1}{x^2 \sqrt{25-x^2}} dx = \int \frac{1}{25 \sin^2 \theta \cdot 5 \cos \theta} \cdot 5 \cos \theta d\theta$$

$$= \frac{1}{25} \int \frac{1}{\sin^2 \theta} d\theta = \frac{1}{25} \int \csc^2 \theta d\theta$$

$$= -\frac{1}{25} \cot \theta + C$$



$$\cot \theta = \frac{\sqrt{25-x^2}}{x}$$

$$\therefore -\frac{1}{25} \frac{\sqrt{25-x^2}}{x} + C$$

(21)

$$\int \frac{x}{\sqrt{x^2+9}} dx$$

$$x = 3 \tan \theta \quad dx = 3 \sec^2 \theta d\theta$$

$$\frac{x}{\sqrt{x^2+9}} = \frac{3 \tan \theta}{\sqrt{9 \tan^2 \theta + 9}} = \frac{3 \tan \theta}{3 \cdot \sec \theta}$$

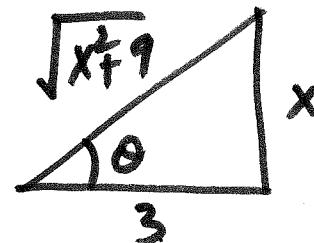
$$\int \frac{x}{\sqrt{x^2+9}} dx = \int \frac{3 \tan \theta}{3 \sec \theta} \cdot 3 \sec^2 \theta d\theta$$

$$= \int 3 \tan \theta \cdot \sec \theta d\theta$$

$$= 3 \sec \theta + C$$

$$= \frac{3 \cdot \sqrt{x^2+9}}{3} + C$$

$$= \sqrt{x^2+9} + C$$



(22)

$$\int \frac{x^2}{\sqrt{x^2-16}} dx$$

$$x = 4 \sec \theta$$

$$dx = 4 \sec \theta \tan \theta d\theta$$

$$\frac{x^2}{\sqrt{x^2-16}} = \frac{16 \sec^2 \theta}{\sqrt{16 \sec^2 \theta - 16}} = \frac{16 \sec^2 \theta}{4 \tan \theta}$$

$$\int \frac{x^2}{\sqrt{x^2-16}} dx = \int \frac{16 \sec^2 \theta}{4 \tan \theta} \cdot 4 \sec \theta \cdot \tan \theta d\theta$$

$$= 16 \int \sec^3 \theta d\theta$$

$$A+B+C=1$$

$$B-C=2$$

$$-A=-1 \quad A=1$$

$$B+C=0$$

$$B-C=2$$

$$2B=2$$

$$B=1 \quad C=1$$

$$\begin{aligned}\int \frac{x^2+2x-1}{x(x-1)(x+1)} dx &= \int \frac{A}{x} dx + \int \frac{B}{x-1} dx + \int \frac{C}{x+1} dx \\ &= \int \frac{1}{x} dx + \int \frac{1}{x-1} dx + \int \frac{1}{x+1} dx \\ &= \ln|x| + \ln|x-1| + \ln|x+1| + C\end{aligned}$$

$$\begin{aligned}
 \int \frac{x^2+2x-1}{x(2x-1)(x+2)} dx &= \int \frac{A}{x} dx + \int \frac{B}{(2x-1)} dx + \int \frac{C}{x+2} dx \\
 &= \int \frac{1}{x} dx + \int \frac{-3/5}{2x-1} dx + \int \frac{-1/5}{x+2} dx \\
 &= \ln|x| - \frac{3}{5} \ln \left| \frac{2x-1}{2} \right| - \frac{1}{5} \ln|x+2| + C
 \end{aligned}$$

(24) $\int \frac{x^2+2x-1}{x^3-x} dx$

$$\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x^2-1)} = \frac{x^2+2x-1}{x(x-1)(x+1)}$$

$$\frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$$

$$\begin{aligned}
 x^2+2x-1 &= A(x-1)(x+1) + Bx \cdot (x+1) + C \cdot x(x-1) \\
 &= A(x^2-1) + Bx^2 + Bx + Cx^2 - CX \\
 &= Ax^2 - A + Bx^2 + Bx + Cx^2 - CX \\
 &= (A+B+C)x^2 + (B-C)x - A
 \end{aligned}$$

(23)

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} dx$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x(2x^2+3x-2)} = \frac{x^2+2x-1}{x(2x-1)(x+2)}$$

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$x^2+2x-1 = A(2x-1)(x+2) + B \cdot x(x+2) + C \cdot x \cdot (2x-1)$$

$$= A \cdot (2x^2+3x-2) + B(x^2+2x) + C(2x^2-x)$$

$$= 2Ax^2 + Bx^2 + 2Cx^2$$

$$+ 3Ax + 2Bx - Cx - 2A$$

$$= (2A+B+2C)x^2 + (3A+2B-C)x - 2A$$

$$-2A = -1 \quad A = 1$$

$$2B - C = -1$$

$$3A + 2B - C = 2$$

$$B + 2C = -1$$

$$2A + B + 2C = 1$$

$$4B - 2C = -2$$

$$\underline{5B = -3} \quad B = -3/5$$

$$-C = -1 - 2\left(-\frac{3}{5}\right)$$

$$-C = -1 + \frac{6}{5}$$

$$C = -\frac{1}{5}$$