000.3 - Mathematical Induction

Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true of all natural numbers (non-negative integers).[1] It is done by proving that the first statement in the infinite sequence of statements is true, and then proving that if any one statement in the infinite sequence of statements is true, then so is the next one.

Mathematical induction should not be misconstrued as a form of inductive reasoning, which is considered non-rigorous in mathematics. In fact, mathematical induction is a form of rigorous deductive reasoning.

1 Introduction

The natural numbers, $\mathbb{N}$, is the set of all non-negative integers:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Very often we want to prove some mathematical statement involving every member of $\mathbb{N}$. The simplest and most common form of mathematical induction proves that a statement involving a natural number n holds for all values of n. The proof consists of two steps:

1. The basis (base case): showing that the statement holds when n is equal to the lowest value that n is given in the question. Usually, $n = 0$ or $n = 1$.

2. The inductive step: showing that if the statement holds for some n, then the statement also holds when $n + 1$ is substituted for n.

The assumption in the inductive step that the statement holds for some n is called the induction hypothesis (or inductive hypothesis). To perform the inductive step, one assumes the induction hypothesis and then uses this assumption to prove the statement for $n+1$.

The choice between $n = 0$ and $n = 1$ in the base case is specific to the context of the proof: If 0 is considered a natural number, as is common in the fields of combinatorics and mathematical logic, then $n = 0$. If, on the other hand, 1 is taken as the first natural number, then the base case is given by $n = 1$.

This method works by first proving the statement is true for a starting value, and then proving that the process used to go from one value to the next is valid. If these are both proven, then any value can be obtained by performing the process repeatedly. It may be helpful to think of the domino effect; if one is presented with a long row of dominoes standing on end, one can be sure that:

1. The first domino will fall.
2. Whenever a domino falls, its next neighbor will also fall.

So it is concluded that all of the dominoes will fall, and that this fact is inevitable. Consider the following statement:

$$0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

for every $n \geq 0$. Suppose that the statement happens to be true for a particular value of n, say $n = k$. Then we have:

$$0 + 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}. \tag{2}$$

We want to start from here, and convince ourselves that the statement is also true for the next value, $n = k + 1$. Plug $k + 1$ into (1),

$$0 + 1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}. \tag{3}$$

Notice that the left-hand side of (3) is the same as the left-hand side of (2) except that there is an extra $k+1$ added to it. So if (2) is true, then we can add $k + 1$ to both sides of it to get:

$$
\begin{aligned}
0+1+2+3+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

showing that (3) is true if (2) is true. To complete the proof, we need only confirm the base case. To do so, simply plug $n = 0$ into the original equation and verify the equality.

2 Formal Definition of Induction

Let $\mathcal{S}(n)$ be any statement about a natural number n. If $\mathcal{S}(0)$ is true and, if $\mathcal{S}(k)$ is true then $\mathcal{S}(k+1)$ is also true, then $\mathcal{S}(n)$ is true for every $n \in \mathbb{N}$.

3 Examples

1. Show that

$$0^2+1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. If $n = 0$ we have the trivial case:

$$0^2 = 0(1)(1)/6.$$

Assume that the equation is true for $n = k$:

$$0^2+1^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}. \tag{4}$$

From (4), we want to show that:

$$
\begin{aligned}
0^2+1^2+\cdots+k^2+(k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}
$$

Begin with (4) and add $(k+1)^2$ to both sides:

$$0^2+1^2+\cdots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

Now just rearrange to complete the proof:

$$0^2 + 1^2 + \cdots + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$\begin{aligned} 0^2 + 1^2 + \cdots + (k+1)^2 &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

2. Let F_k be the Fibonacci numbers defined by: $F_0 = 0$, $F_1 = 1$, and if $k > 1$, $F_k = F_{k-1} + F_{k-2}$. Show that:

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n$$

and that

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

Solution. Part 1:
First check for $n = 1$:

$$F_0 F_2 = 0 \cdot 2 = 0 = F_1^2 + (-1)^1 = 1 - 1 = 0.$$

If we assume it is true for $n = k$, we have:

$$F_{k-1}F_{k+1} = F_k^2 + (-1)^k. \tag{5}$$

From (5), we need to show that the equality continues to hold for $n = k + 1$, i.e., we need to show that if we start with (5) we can show that:

$$F_k F_{k+2} = F_{k+1}^2 + (-1)^{k+1}.$$

Since $F_{k+2} = F_k + F_{k+1}$, the equation above is equivalent to:

$$F_k(F_k + F_{k+1}) = F_{k+1}^2 + (-1)^{k+1}.$$

or to

$$F_k^2 + F_k F_{k+1} = F_{k+1}^2 + (-1)^{k+1}.$$

Substitute F_k^2 from the right-hand side of (5):

$$F_{k-1}F_{k+1} - (-1)^k + F_k F_{k+1} = F_{k+1}^2 + (-1)^{k+1}$$

or

$$F_{k+1}(F_{k-1}+F_k) = F_{k+1}^2 + (-1)^{k+1} + (-1)^k = F_{k+1}^2$$

or

$$F_{k+1}^2 = F_{k+1}^2$$

Part 2:
For $n = 0$:

$$\sum_{i=0}^{0} F_i^2 = F_0^2 = 0 = F_0 F_1 = 0 \cdot 1 = 0$$

If it is true for $n = k$:

$$\sum_{i=0}^{k} F_i^2 = F_k F_{k+1} \tag{6}$$

then we can F_{k+1}^2 to both sides of (6) to get:

$$\sum_{i=0}^{k+1} F_i^2 = F_k F_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1} F_{k+2}$$

3. Show that:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}.$$

Solution. For the $n = 1$ case we have

$$1 \leq 2\sqrt{1} = 2.$$

Assume the equation is true for $n = k$:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \leq 2\sqrt{k}.$$

To show it is also true for $n = k + 1$, add $1/\sqrt{k+1}$ to both sides:

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

Thus if we can show that

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}$$

then the proof is complete. Multiply both sides by $\sqrt{k+1}$ and then square both sides to obtain:

$$4k(k+1) + 4\sqrt{k(k+1)} + 1 \le 4(k^2 + 2k + 1).$$

$$4\sqrt{k(k+1)} \le 4k + 3$$

Squaring again:

$$16k^2 + 16k \le 16k^2 + 24k + 9$$

which is always true.

4. Show that:

$$2! \cdot 4! \cdot 6! \cdots (2n)! \ge ((n+1)!)^n.$$

Solution. First show it is true for $n = 1$:

$$2! = 2 \ge (2!)^1 = 2.$$

Next assume it is true for $n = k$:

$$2! \cdot 4! \cdot 6! \cdots (2k)! \ge ((k+1)!)^k. \tag{7}$$

If we multiply both sides of (7) by $(2(k+1))!$, we get:

$$2! \cdot 4! \cdots (2k)! \cdot (2k+2)! \ge ((k+1)!)^k \cdot (2k+2)!.$$

If it can be shown that the right-hand side of the equation above is larger than $((k+2)!)^{k+1}$, the proof is complete. The term $(2k+2)!$ on the right-hand side can be written:

$$(2k+2)! = (2k+2)(2k+1)(2k) \cdots (k+3)(k+2)!$$

This consists of k terms, all greater than $k+2$, multiplied by $(k+2)!$, so

$$\begin{aligned} ((k+1)!)^k(2k+2)! &> ((k+1)!)^k(k+2)^k(k+2)! \\ &= ((k+2)!)^k(k+2)! = ((k+2)!)^{k+1} \end{aligned}$$

5. Show that:

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = 2\cos\frac{\pi}{2^{n+1}}$$

where there are n 2s in the expression on the left.
Solution. For $n=1$ case we have:

$$\sqrt{2} = 2\cos\frac{\pi}{2^2} = 2\cos\pi/4 = 2\sqrt{2}/2 = \sqrt{2}.$$

Now assume it is true for k nested square roots:

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = 2\cos\frac{\pi}{2^{k+1}}.$$

Add 2 to both sides and take the square root, so the LHS will have $k+1$ nested square roots, and the right hand side will be:

$$\sqrt{2+2\cos\frac{\pi}{2^{k+1}}}. \tag{8}$$

All we need to show is that the value above is equal to

$$2\cos\frac{\pi}{2^{k+2}}. \tag{9}$$

We know that for any angle θ we have:

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}. \tag{10}$$

Substitute $\pi/2^{k+2}$ for θ in 10 and the equality of 8 and 9 becomes immediately obvious.

4 References

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