000.1 - The Law of Cosines and Sines

The law of cosines and the law of sines are two trigonometric equations commonly applied to find lengths and angles in a general triangle.[1]

1 The Law of Cosines

Suppose we have a triangle labeled by the vertex points $A, B,$ and C, sides a, b and c, and angles $\alpha, \beta,$ and γ, as shown in Figure 1. In what follows, we consider all lengths and angles to be positive quantities.

The law of cosines is used to solve a triangle for

- a third side, c, of the triangle, if two sides, a and b, and the angle γ between them is known:

$$c = \sqrt{a^2 + b^2 - 2ab\cos\gamma}$$

- the angles α, β and γ if the three sides, a, b and c are known:

$$\begin{aligned}\alpha &= \arccos\frac{b^2 + c^2 - a^2}{2bc}\\ \beta &= \arccos\frac{a^2 + c^2 - b^2}{2ac}\\ \gamma &= \arccos\frac{a^2 + b^2 - c^2}{2ab}\end{aligned}$$

- the third side a of a triangle if one knows two sides, say, b and c, and an angle opposite one of them, say, γ:

$$a = b\cos\gamma \pm \sqrt{c^2 - b^2\sin^2\gamma}$$

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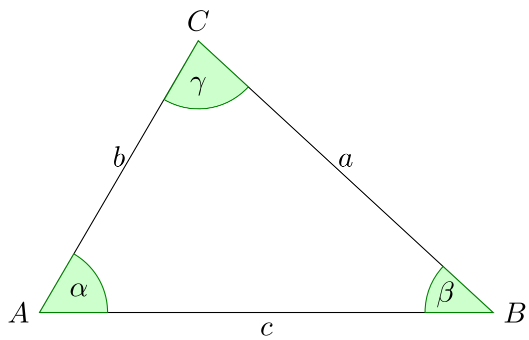

Figure 1: An arbitrary triangle in Euclidean 2-space.

This last equation can have 2, 1, or 0 positive solutions corresponding to the number of possible triangles fulfilling the equation. It will have two positive solutions if $b\sin\gamma < c < b$, only one positive solution if $c \geq b$ or $c = b\sin\gamma$, and no solution if $c < b\sin\gamma$. It can be understood with the help of the following relationship between sin and arccos:

Consider a right triangle, such that for the acute angle θ, $\cos\theta = x$ (x is positive and the length of the hypotenuse is 1), then $\theta = \arccos x$. The length of the side opposite θ is found using the Pythagorean theorem; its length is $\sqrt{1-x^2}$ since $x^2 + (\sqrt{1-x^2})^2 = 1$. It follows that:

$$\sin(\arccos x) = \sin\theta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Now we will use this to help prove the third formula from the list. Begin by substituting the second equation from the list into the third and use our derived identity as well:

$$b\frac{a^2+b^2-c^2}{2ab} \pm \sqrt{c^2 - b^2\sin^2\left(\arccos\left(\frac{a^2+b^2-c^2}{2ab}\right)\right)}$$

$$\frac{a^2+b^2-c^2}{2a} \pm \sqrt{c^2 - b^2\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right]}$$

$$\frac{a^2+b^2-c^2}{2a} \pm \sqrt{c^2 - b^2 + \left[\frac{a^2+b^2-c^2}{2a}\right]^2}$$

$$\frac{a^2+b^2-c^2}{2a} \pm \sqrt{\frac{4a^2(c^2-b^2)}{4a^2} + \frac{a^4+2a^2b^2+b^4-2a^2c^2-2b^2c^2+c^4}{4a^2}}$$

$$\frac{a^2+b^2-c^2}{2a} \pm \frac{1}{2}\sqrt{\frac{2a^2c^2-2a^2b^2+a^4+b^4-2b^2c^2+c^4}{a^2}}$$

$$\frac{a^2+b^2-c^2 \pm a\sqrt{\frac{(a^2-b^2+c^2)^2}{a^2}}}{2a}$$

From this point, we must handle the $(+)$- and $(-)$-cases separately. The $(+)$-case obviously evaluates to a. To understand the $(-)$-case, we take note that we must have $b^2-c^2=a^2$, i.e., $a^2+c^2=b^2$, in order for the following equation to equal a.

$$\frac{b^2-c^2}{a} \Rightarrow b^2-c^2=a^2 \Rightarrow a^2+c^2=b^2$$

So we see that the plus and minus sign alternatives take into account the fact that we must have either $a^2+b^2=c^2$ or $a^2+c^2=b^2$ (in the case of a right triangle), and the law of cosines is thus a generalization of the pythagorean theorem for general Euclidean triangles (applies to triangles that are not right triangles in Euclidean space).

Theorem-Law of Cosines. *Suppose we have a triangle labeled by the vertex points $A, B,$ and C, sides a, b and c, and angles $\alpha, \beta,$ and γ, as shown in Figure 1, and consider all lengths and angles to be positive quantities. Further suppose that two sides, a and b, and the angle γ between them is known. Then a third side, c, of the triangle can be determined as*

$$c = \sqrt{a^2+b^2-2ab\cos\gamma}$$

Proof of the Law of Cosines. Now, we use the distance formula to determine the length of a^2 (see Figure 2).

$$\begin{aligned} a^2 &= (b\cos\alpha - c)^2 + (b\sin\alpha - 0)^2 \\ &= b^2\cos^2\alpha - 2bc\cos\alpha + c^2 + b^2\sin^2\alpha \\ &= b^2(\cos^2\alpha + \sin^2\alpha) + c^2 - 2bc\cos\alpha \\ &= b^2 + c^2 - 2bc\cos\alpha \end{aligned}$$

Since the other two cases follow from a simple relabeling of the triangle, the proof is complete. □

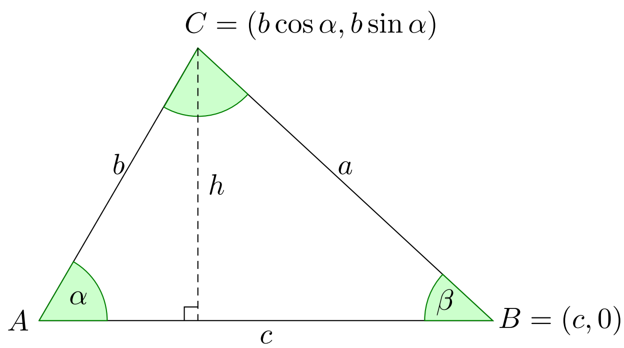

Figure 2: Construction for proving the law of cosines , as well as the law of sines.

2 The Law of Sines

The law of sines is a formula stating a rule of proportionality between the lengths of the sides of an arbitrary triangle and the sines of its angles. The law of sines can be used to perform "triangulation" – computing the remaining sides of a triangle when two angles and a side are known.

Theorem-Law of Sines. *Suppose we have a triangle labeled by the vertex points $A, B,$ and $C,$ sides a, b and $c,$ and angles $\alpha, \beta,$ and $\gamma,$ as shown in Figure 1, and consider all lengths and angles to be positive quantities. Then*

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Proof of the Law of Sines. Make a triangle with the sides a, b, and c, and angles α, β, and γ. Draw the altitude from vertex C to the side across c; by definition it divides the original triangle into two right angle triangles. Mark the length of this line h (see Figure 2). It is easily observed that

$$\sin\alpha = \frac{h}{b} \qquad , \qquad \sin\beta = \frac{h}{a}$$

Thus

$$h = b\sin\alpha = a\sin\beta$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

If the same procedure is followed between vertex A and side a you find:

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

□

3 References

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7. R. Larson, *Algebra and Trigonometry*, Brooks Cole; 5th edition, 2000.