000.2 - The Triangle Inequality

The Cauchy-Schwarz inequality and the triangle inequality have many theoretical and practical applications including, but not limited to, normed vector spaces, inner product spaces, p-norms and metric spaces.[1] We shall see that the Cauchy-Schwarz inequality is helpful in a proof of the triangle inequality.

In its most basic form, the triangle inequality states that for any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side. In Euclidean geometry and some other geometries the triangle inequality is a theorem about distances. For Euclidean right triangles it is a consequence of Pythagoras' theorem, and for general Euclidean triangles a consequence of the law of cosines, although it may optionally be proven without these theorems.

1 Derivations and Proofs

Lemma. *Suppose we have a triangle labeled by the vertex points O, A, and B, sides $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, and angle θ, as shown in Figure 1, and consider all lengths and angles to be positive quantities. Then*

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \cos\theta$$

Proof of the Dot Product Formula. Given two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, the dot product is **defined** as

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Note that for any particular vector $\mathbf{a}$, we have

$$\mathbf{a} \cdot \mathbf{a} = a_1a_1 + a_2a_2 + a_3a_3 = ||\mathbf{a}||^2$$

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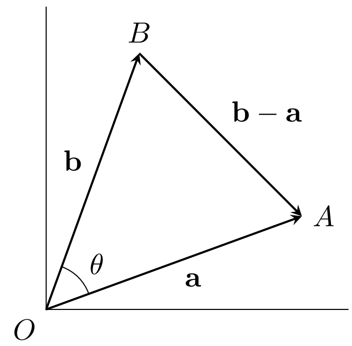

Figure 1: The three vectors above form the triangle AOB and note that the length of each side is nothing more than the magnitude of the vector forming that side.

Now the law of cosines says,

$$||\mathbf{a}-\mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2\,||\mathbf{a}||\,||\mathbf{b}||\cos\theta$$

We can write the left-hand side as,

$$\begin{aligned} ||\mathbf{a}-\mathbf{b}||^2 &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} \\ &= ||\mathbf{a}||^2 - 2\mathbf{a}\cdot\mathbf{b} + ||\mathbf{b}||^2 \end{aligned}$$

So we arrive at,

$$\begin{aligned} ||\mathbf{a}-\mathbf{b}||^2 &= ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2\,||\mathbf{a}||\,||\mathbf{b}||\cos\theta \\ ||\mathbf{a}||^2 - 2\mathbf{a}\cdot\mathbf{b} + ||\mathbf{b}||^2 &= ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2\,||\mathbf{a}||\,||\mathbf{b}||\cos\theta \\ -2\mathbf{a}\cdot\mathbf{b} &= -2\,||\mathbf{a}||\,||\mathbf{b}||\cos\theta \\ \mathbf{a}\cdot\mathbf{b} &= ||\mathbf{a}||\,||\mathbf{b}||\cos\theta \end{aligned}$$

□

Theorem. (Cauchy-Schwarz Inequality). *For any vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, both in* $\mathbf{R}^2$ *or both in* $\mathbf{R}^3$*,*

$$|\mathbf{u}\cdot\mathbf{v}| \leq ||\mathbf{u}||\,||\mathbf{v}||\,. \tag{1}$$

The absolute value of the scalar product of two vectors is less than or equal to the product of their norms.

Proof of the Cauchy-Schwarz Inequality. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then (1) just says $0 \leq 0$, which is true. Next suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors. Then, as shown previously,

$$\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\cos\theta$$

where θ is the angle between the vectors. Taking absolute values and using the fact that $|\cos\theta| \leq 1$,

$$|\mathbf{u}\cdot\mathbf{v}| = ||\mathbf{u}||\,||\mathbf{v}||\,|\cos\theta| \leq ||\mathbf{u}||\,||\mathbf{v}||$$

□

Theorem. (Triangle Inequality). *For any vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, both in* $\mathbf{R}^2$ *or both in* $\mathbf{R}^3$*,*

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||\,. \tag{2}$$

The norm of the sum of two vectors is less than or equal to the sum of their norms.

Proof of the Cauchy-Schwarz Inequality. Since both sides of (2) are nonnegative, (1) is equivalent to (meaning (1)and (2) imply each other):

$$(||\mathbf{u}+\mathbf{v}||)^2 \leq (||\mathbf{u}|| + ||\mathbf{v}||)^2.$$

So we only need to show this to complete the proof.

$$\begin{aligned}
(||\mathbf{u}+\mathbf{v}||)^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\
&= ||\mathbf{u}||^2 + 2\mathbf{u}\cdot\mathbf{v} + ||\mathbf{v}||^2 \\
&\leq ||\mathbf{u}||^2 + 2|\mathbf{u}\cdot\mathbf{v}| + ||\mathbf{v}||^2 \\
\text{(by Cauchy-Schwarz)} \quad &\leq ||\mathbf{u}||^2 + 2\,||\mathbf{u}||\,||\mathbf{v}|| + ||\mathbf{v}||^2 \\
&= (||\mathbf{u}|| + ||\mathbf{v}||)^2
\end{aligned}$$

□

A generalization of the triangle inequality is

$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|$$

2 References

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