Solutions to mock final, Discrete Structures Spring semester 2011

The final exam will consist of at most four regular assignments and at most eight multiple choice ones for a total of t assignments. The assignments will be similar to the ones below, but there may be at most one assignment different from the ones below.

Per assignment you can get at most one point, for a total of at most t points. If you hand in all your exam sheets, your grade is calculated as $1 + \frac{5}{t}s$ rounded in the usual fashion to the nearest half point, where s is the total number of points you obtained; otherwise, if you don't hand them in, your grade will be zero.

Several of the multiple choice assigments consist of a number of subquestions, each of these subquestions with two or more possible choices just one of which is correct and just one of which must be ticked: if there are m subquestions, the correct choice is worth $\frac{1}{m}$ point, but no points are given if no circle or more than one circle is ticked (per subquestion).

For each of the other multiple choice assignments there is one correct answer. If you tick just the correct choice, you get one point; otherwise (if you tick no circle or more than one circle) you get zero points.

Note that the argumentation given below for the multiple choice assignments is for your explanation only: for the multiple choice assignments in the final exam, no argumentation needs to be given.

1

Let $p = 1009$ and $x = 78$. Assuming that $\{0, 1, 2, \ldots, p-1\}$ is used as the set of representatives for the residue classes modulo p, the representative for the residue class of the multiplicative inverse $x^{-1} \bmod p$ of x modulo p belongs to

○ $\{0, 1, 2, \ldots, \lfloor p/4 \rfloor\}$;

○ $\{\lfloor p/4 \rfloor + 1, \lfloor p/4 \rfloor + 2, ..., \lfloor p/2 \rfloor\}$;

✓ $\{\lfloor p/2 \rfloor + 1, \lfloor p/2 \rfloor + 2, ..., 3\lfloor p/4 \rfloor\}$;

○ $\{3\lfloor p/4 \rfloor + 1, 3\lfloor p/4 \rfloor + 2, ..., p-1\}$.

$$\begin{aligned}
0 \cdot 78 &\equiv 1009 \pmod{1009} \\
1 \cdot 78 &\equiv 78 \pmod{1009} \text{ (subtract 13 times from previous)} \\
-13 \cdot 78 &\equiv -5 \pmod{1009} \text{ (add 16 times to previous)} \\
-207 \cdot 78 &\equiv -2 \pmod{1009} \text{ (subtract 3 times from previous)} \\
608 \cdot 78 &\equiv 1 \pmod{1009}
\end{aligned}$$

Because $608 \in \{\lfloor p/2 \rfloor + 1, \lfloor p/2 \rfloor + 2, ..., 3\lfloor p/4 \rfloor\}$ the third circle must be checked.

2

For each statement below tick only the circle that applies.

1. For different constants $a > 1$ and $b > 1$ and a constant positive integer k, the function $(log_a(n))^k$ is $\theta((log_b(n))^k)$.

   ✓ This is correct.

   ○ This is incorrect.

   From $\log_a(n) = \log_b(a) \cdot \log_b(n)$ it follows that $(log_a(n))^k = C \cdot (log_b(n))^k$ where C is the non-zero constant value $(\log_b(a))^k$. Thus $(log_a(n))^k$ is $O((log_b(n))^k)$ and $(log_b(n))^k$ is $O((log_a(n))^k)$, so that $(log_a(n))^k$ is $\theta((log_b(n))^k)$ and the first circle must be ticked.

2. For different constants $a > 1$ and $b > 1$ and a constant $k > 1$, the function $n^{log_a(k))}$ is $\theta(n^{log_b(k)})$.

   ○ This is correct.

   ✓ This is incorrect.

   Let $c_1 = \log_a(k)$ and $c_2 = \log_b(k)$ be constants. Because $a \neq b$ it follows that $c_1 \neq c_2$. Without loss of generality, assume that $c_1 > c_2$. It follows that $c_1 - c_2 > 0$ and that $n^{log_a(k))}/n^{log_b(k)} = n^{c_1 - c_2}$ is not bounded by a constant. Thus, $n^{log_a(k))}$ is not $O(n^{log_b(k)})$ and $n^{log_a(k))}$ is not $\theta(n^{log_b(k)})$, so the second circle must be ticked.

3. The function n^n is $O(n!)$.

   ○ This is correct.

✓ This is incorrect.

Because $n! < n^{n/2} \cdot \left(\frac{n}{2}\right)^{n/2}$ it follows that $n^n/n! > 2^{n/2}$ so that n^n is not $O(n!)$ and the second circle must be ticked.

4. The function $n\log(n)$ is $O(\log(n!))$.

   ✓ This is correct.

   ◯ This is incorrect.

   See the 13th slide of the 8th lecture given on March 16 why the first circle must be ticked.

5. The function $f : \mathbf{R} \to \mathbf{R}$ that maps $x \in \mathbf{R}$ to x^3 is a bijection.

   ✓ This is correct.

   ◯ This is incorrect.

   See exercise 2.3.19.c. To convince yourself that f is indeed a bijection, you may for instance draw a graph of f (cf. page 142-143; of course it helps to know the definition of a bijection).

6. The function $g : \mathbf{Z}_{\geq 2} \to \mathbf{Z}_{\geq 1}$ that maps n to the number of distinct prime divisors of n is a surjection.

   ✓ This is correct.

   ◯ This is incorrect.

   To show that this is a surjection, we need to show that for any target-value $t \in \mathbf{Z}_{\geq 1}$ there is at least one integer $n \in \mathbf{Z}_{\geq 2}$ such that $g(n) = t$. Denote by p_i for $i \in \mathbf{Z}_{>0}$ the ith prime number. Then $n = \prod_{i=1}^{t} p_i$ satisfies $g(n) = t$.

3

There are 4 men (m_1, m_2, m_3, m_4) and 4 women (w_1, w_2, w_3, w_4), with the following preference lists:

m_1: w_1, w_2, w_3, w_4; w_1: m_3, m_1, m_2, m_4;

m_2: w_1, w_4, w_2, w_3; w_2: m_1, m_3, m_4, m_2;

m_3: w_4, w_1, w_3, w_2; w_3: m_4, m_1, m_2, m_3;

m_4: w_3, w_4, w_2, w_1; w_4: m_3, m_4, m_1, m_2.

Consider the matching (m_1, w_1), (m_2, w_2), (m_3, w_4), (m_4, w_3). Tick the circle that applies.

◯ The matching is not stable.

◯ The matching is stable and male optimal.

◯ The matching is stable and female optimal.

◯ The matching is stable, not male optimal, and not female optimal.

✓ The matching is stable and both male and female optimal.

Use the stable marriage algorithm as presented in class on June 1 (see the 16th slide of the 26th lecture) to find the male optimal stable matching:

- m_1 proposes to his top choice w_1. Because w_1 is not matched to anyone yet, she accepts and the pair (m_1, w_1) is formed.
- m_2 proposes to his top choice w_1. Because w_1 is already matched to m_1, she compares her current proposal by m_2 to her current partner m_1: because she prefers m_1 to m_2, she declines m_2's proposal and w_1 is removed from m_2's preference list.
- m_2 proposes to w_4, his new top-choice (after w_1 got removed from m_2's list). Because w_4 is not matched to anyone yet, she accepts and the pair (m_2, w_4) is formed.
- m_3 proposes to his top-choice w_4. Because w_4 is already matched to m_2, she compares her current proposal by m_3 to her current partner m_2: because she prefers m_3 to m_2, she breaks up with m_2 and accepts m_3's proposal. The pair (m_3, w_4) is formed and w_4 is removed from m_2's preference list.
- m_2 proposes to w_2, his new top-choice (after also w_4 got removed from m_2's list). Because w_2 is not matched to anyone yet, she accepts and the pair (m_2, w_2) is formed.
- m_4 proposes to his top-choice w_3. Because w_3 is not matched to anyone yet, she accepts and the pair (m_4, w_3) is formed.
- At this point all men (and thus all women) are matched: and the resulting matching (m_1, w_1), (m_2, w_2), (m_3, w_4), (m_4, w_3) is stable and male optimal.

Using the algorithm again, but now with the women proposing to find a female optimal matching, we find the following:

- w_1 proposes to m_3: form pair (m_3, w_1).
- w_2 proposes to m_1: form pair (m_1, w_2).
- w_3 proposes to m_4: form pair (m_4, w_3).
- w_4 proposes to m_3: m_3 prefers w_4 to w_1 so (m_3, w_1) split up, form pair (m_3, w_4), remove m_3 from w_1's list.
- w_1 proposes to m_1: m_1 prefers w_1 to w_2 so (m_1, w_2) split up, form pair (m_1, w_1), remove m_1 from w_2's list.
- w_2 proposes to m_3: m_3 prefers w_4 to w_2, remove m_3 from w_2's list.
- w_2 proposes to m_4: m_4 prefers w_3 to w_2, remove m_4 from w_2's list.
- w_2 proposes to m_2: form pair (m_2, w_2).

This results in the female optimal stable matching (m_4, w_3), (m_3, w_4), (m_1, w_1), (m_2, w_2). Because the female optimal stable matching thus found is the same as the male optimal stable matching found earlier, the fourth circle must be ticked.

4

Prove the "if"-part of the following statement.

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

The "if"-part of the statement is "$\leftarrow$"; thus we assume that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$ and we need to prove that R is transitive.

To prove that R is transitive, we need to prove that if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. If $(a, b) \in R$ and $(b, c) \in R$, then according to the definition of $R^2 = R \circ R$ and the definition of the composition operation "$\circ$" on relations, it follows that $(a, c) \in R^2$. According to our assumption it is the case that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$; applying this for $n = 2$ we find that $R^2 \subseteq R$. Thus, if $(a, c) \in R^2$, then $(a, c) \in R$, which is what we had to prove.

5

For each subquestion below tick only the circle that applies.

1. The compound proposition $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is

   ✓ a tautology;

   ◯ a contingency;

   ◯ a contradiction.

   To check that this is indeed a tautology, you can use a truth table. See also the last row of Table 1 on page 66 and the paragraphs on "Resolution" on pages 68 and 69.

2. The compound proposition $((p \rightarrow r) \wedge (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$ is

   ◯ a tautology;

   ✓ a contingency;

   ◯ a contradiction.

   It is true if $p = q = r =$ "true" (and therefore not a contradiction) but false if $p =$ "true", $q =$ "false", and $r =$ "false" (and therefore not a tautology). Because it is neither a tautology nor a contradiction, it is a contingency.

3. The compound proposition $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ is

   ✓ a tautology;

   ◯ a contingency;

   ◯ a contradiction.

To check that this is indeed a tautology, you can use a truth table. See also the first row of Table 7 on page 25.

4. The compound proposition $(p \wedge q) \leftrightarrow (p \rightarrow \neg q)$ is

   ◯ a tautology;

   ◯ a contingency;

   ✓ a contradiction.

   To check that this is indeed a contradiction, you can use a truth table. See also the fourth row of Table 7 on page 25.

5. The statement $\forall x \exists y((x < y) \rightarrow (x^2 < y^2))$, where the domain of discourse is $\mathbf{R}$ for both x and y, is

   ✓ true;

   ◯ false.

   For any $x \in \mathbf{R}$ one can pick $y \in \mathbf{R}$ such that $x < y$ = "false" (for instance, one may pick $y = x - 1$). Because the proposition "false"$\rightarrow r$ is true for any proposition r, the statement is correct.

6. The statement $\forall x \forall y(x^2 < y + 1)$, where the domain of discourse is $\mathbf{R}$ for both x and y, is

   ◯ true;

   ✓ false.

   For $x = 1$ and $y = 0$ the statement $x^2 < y+1$ is false, thus $\exists x \exists y(x^2 \geq y+1)$ is true, which is the negation of the statement $\forall x \forall y(x^2 < y + 1)$, which is therefore false.

6

If n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Prove the above statement in the following two ways:

1. using a combinatorial proof;

   Suppose that given a group of n persons a group of k including a leader needs to be selected.

   We may first select the group of k (there are $\binom{n}{k}$ ways to do so) and then select the leader from among the group of k. With the product rule it follows that there are $k\binom{n}{k}$ ways to select our group of k with a leader.

   Alternatively, we may first select the group leader from among the n persons (n ways to do so), after which we select the remaining $k - 1$ group members from the remaining $n - 1$ persons ($\binom{n-1}{k-1}$ ways to do so). With the product rule it follows that there are $n\binom{n-1}{k-1}$ ways to select our group of k with a leader.

   It follows that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

2. using an algebraic proof.

$$\begin{aligned} k\binom{n}{k} &= \frac{k \cdot n!}{k!(n-k)!} \\ &= \frac{n \cdot (n-1)!}{(k-1)!((n-1)-(k-1))!} \\ &= n\binom{n-1}{k-1} \end{aligned}$$

7

Consider the arithmetic expression

$$((a*b)-(c^2+3*d*e))/(4-f^5)$$

and the binary tree associated with it.

1. The infix form of the arithmetic expression is

   ◯ $((a*b)-(c\hat{}2+3*d*e))/(4-f\hat{}5)$
   ✓ $(((a*b)-((c\hat{}2)+(3*(d*e))))/(4-(f\hat{}5)))$.

   According to the definition on page 719, the infix form is fully parenthesized: it includes all open and close parentheses that are included when going down a left branch and going up a right branch, respectively (see slide 3 of 26th lecture on June 1). The first expression is not fully parenthesized.

2. a prefix form of the arithmetic expression is

   ◯ $/ - *a\,b + *3\,d * e\hat{}\,c\,2 - 4\hat{}\,f\,5$
   ✓ $/ - *a\,b + \hat{}\,c\,2 * 3 * d\,e - 4\hat{}\,f\,5$

   The first expression is a prefix form of $((a*b)-(3*d+e*c^2))/(4-f^5)$ and therefore incorrect.

3. a postfix form of the arithmetic expression is

   ◯ $a\,b * c\,2\hat{}\,3\,d\,e * * + 4 - f\,5\hat{} - /$
   ✓ $a\,b * 3\,d * e * c\,2\hat{} + -4\,f\,5\hat{} - /$

   The first expression is a postfix form of $(a*b)/(c^2+3*d*e-4-f^5)$ and therefore incorrect.

4. A breadth-first search of the binary tree, starting at the node labelled "3" visits the nodes in the following order:

   ◯ $\{/\},\{-,-\},\{*,+,4,\hat{}\},\{a,b,\hat{},*,f,5\},\{c,2,3,*\},\{d,e\}$
   ✓ $\{3\},\{*\},\{+,*\},\{-,\hat{},d,e\},\{/,*,c,2\},\{-,a,b\},\{4,\hat{}\},\{f,5\}$
   ◯ $\{3\},\{*\},\{+,*\},\{-,\hat{},d,e\},\{*,c,2\},\{/,a,b\},\{-\},\{4,\hat{}\},\{f,5\}$

The first possibility corresponds to a breadth-first search of the binary tree starting at the root node labelled "/". The last possibility corresponds to a breadth-first search of a tree where the node labelled "/" is linked to a node labelled "$*$", to the node labelled "c", or to the node labelled "2", and can therefore not be correct.

8

For a random variable T on a sample space we denote by $E(T)$ its expected value and by $V(T)$ its variance.

Let X and Y be random variables on the same sample space. For each statement below tick only the circle that applies.

1. $E(X+Y) = E(X) + E(Y)$.

   ✓ This is correct.

   ◯ This is incorrect.

   See Theorem 3 on page 429.

2. If X and Y are independent, then $V(X+Y) = V(X) + V(Y)$.

   ✓ This is correct.

   ◯ This is incorrect.

   See Theorem 7 on page 437.

3. If $p(Y) = \frac{1}{2}$, $p(X|Y) = \frac{2}{5}$, and $p(Y|X) = \frac{3}{5}$, then

   ✓ $p(X) = \frac{1}{3}$;

   ◯ $p(X) = \frac{3}{4}$.

   According to the definition of conditional probability on page 404 (and assuming the relevant probabilities are positive), $p(X|Y) = \frac{p(X\cap Y)}{p(Y)}$ and $p(Y|X) = \frac{p(Y\cap X)}{p(X)}$. It follows that $p(X|Y)p(Y) = p(Y|X)p(X)$ and thus that $p(X) = \frac{p(X|Y)p(Y)}{p(Y|X)} = \frac{(2/5)(1/2)}{(3/5)} = \frac{1}{3}$.

9

For each subquestion below tick only the correct circle.

1. Given a regular deck of 52 cards, the probability that a randomly chosen hand of five cards in poker contains two pairs is

   ◯ $\binom{13}{2}\binom{4}{2}^2\binom{48}{1}/\binom{52}{5}$;

   ◯ $\binom{13}{2}\binom{4}{2}\binom{48}{1}/\binom{52}{5}$;

   ✓ $\binom{13}{2}\binom{4}{2}^2\binom{44}{1}/\binom{52}{5}$.

Pick the two different kinds (in $\binom{13}{2}$ ways), pick the two suits for the first kind (in $\binom{4}{2}$ ways), pick the two suits for the second kind (in $\binom{4}{2}$ ways), and pick the remaining card while avoiding the four cards already picked and the two plus two remaining cards of the two kinds picked (in $52-4-2-2=44=\binom{44}{1}$ ways). With the product rule it follows that there are $\binom{13}{2}\binom{4}{2}^2\binom{44}{1}$ different hands of five cards that contain two pairs. Since there are $\binom{52}{4}$ different hands of five cards of poker, it follows that the third circle must be ticked.

2. The number of integer solutions $x_1, x_2, x_3 \geq 0$ of $x_1 + x_2 + x_3 = 6$ is

   ✓ 28;

   ○ 10;

   ○ 84.

   According to Example 5 on pages 373-374, there are $C(3+6-1, 6) = \binom{8}{6} = 28$ solutions: the first circle must be ticked.

3. Given a rectangular mesh of horizontal streets numbers $0, 1, 2, 3, ...$ and vertical avenues named A, B, C, D,..., the number of ways to go from the corner of 36th street and Avenue E to the corner of 42nd street and Avenue A (in the shortest possible way) is

   ○ $\binom{6}{4}$;

   ✓ $\binom{10}{4}$;

   ○ 1.

   The walk needs to move up $42-36=6$ streets and across four avenues (E→D→C→B→A). Thus the walk consist of a total of ten blocks, four of which are avenue blocks, so that the correct answer is $\binom{10}{4}$: the second circle must be ticked (see also slide 3 of 25th lecture on May 31).

10

Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

Let $G(x) = \sum_{i=0}^{\infty} a_i x^i$. Then $G(x) = a_0 + \sum_{i=1}^{\infty} a_i x^i$, and thus $G(x) = 1 + x\sum_{j=0}^{\infty} a_{j+1} x^j$ because $a_0 = 1$. With $a_{j+1} = 3a_j + 2$ we find $G(x) = 1 + x\sum_{j=0}^{\infty}(3a_j + 2)x^j$ and $G(x) = 1 + 3x\sum_{j=0}^{\infty} a_j x^j + 2x\sum_{j=0}^{\infty} x^j$. With $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ (cf. fifth row of Table 1 on page 489) it follows that

$$G(x) = 1 + 3xG(x) + \frac{2x}{1-x}$$

and thus that

$$G(x)(1-3x) = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}.$$

Writing

$$G(x) = \frac{1+x}{(1-x)(1-3x)}$$

as

$$G(x) = \frac{u}{1-3x} + \frac{v}{1-x}$$

and solving for u and v, we find that $u+v=1$ and $-u-3v=1$ so that $u=2$ and $v=-1$ and thus

$$G(x) = \frac{2}{1-3x} - \frac{1}{1-x}.$$

Using $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ again it follows that $G(x) = 2\sum_{i=0}^{\infty}(3x)^i - \sum_{i=0}^{\infty} x^i$ and thus that $G(x) = \sum_{i=0}^{\infty}(2 \cdot 3^i - 1)x^i$. It follows that $a_i = 2 \cdot 3^i - 1$.

11

Use Dijkstra's algorithm to find all shortest paths from the vertex a to all other vertices in the weighted graph below.

At each step of the algorithm clearly indicate all label values and how they change in the course of the algorithm. Clearly list all resulting shortest path values.

Initially $L(a) = 0$, $S =$, $L(b) = L(c) = \ldots = L(l) = \infty$.

1. The smallest label value for vertices not in S is obtained for a. Thus, add a to S and check which of the vertices not in S that are connected via an edge to a (namely, vertices b, c, and d) may get lower label values. As a result we find: $S = \{a\}$ with shortest path value $L(a) = 0$; and we need to update the labels of the following vertices not in S: $L(b) = \min(\infty, 4) = 4$, $L(c) = \min(\infty, 3) = 3$, $L(d) = \min(\infty, 6) = 6$ (the other labels remain unchanged).

   The shortest path from a to a has weight zero and is empty.

2. The smallest label value for vertices not in S is obtained for c. Thus, add c to S and check which of the vertices not in S that are connected via an edge to c (namely, vertices d, f, and h) may get lower label values. As a result we find: $S = \{a, c\}$ with shortest path values $L(a) = 0$, $L(c) = 3$; and we need to update the labels of the following vertices not in S: $L(d) = \min(6, 3+1) = 4$, $L(f) = \min(\infty, 3+5) = 8$, $L(h) = \min(\infty, 3+4) = 7$ (the other labels remain unchanged).

   The shortest path from a to c has weight 3 and consists of the edge $\{a, c\}$.

3. A smallest label value for vertices not in S is obtained for b. Thus, add b to S and check which of the vertices not in S that are connected via an edge to b (namely, vertices d, e, and g) may get lower label values. As a result we find: $S = \{a, b, c\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$; and we need to update the labels of the following vertices not in S: $L(d) = \min(4, 4+5) = 4$, $L(e) = \min(\infty, 4+6) = 10$, $L(g) = \min(\infty, 4+3) = 7$ (the other labels remain unchanged).

   The shortest path from a to b has weight 4 and consists of the edge $\{a, b\}$.

4. The smallest label value for vertices not in S is obtained for d. Thus, add d to S and check which of the vertices not in S that are connected via an edge to d (namely, vertices f and g) may get lower label values. As a result we find: $S = \{a, b, c, d\}$ with shortest path values $L(a) = 0$, $L(b) = 4$,

$L(c) = 3$, $L(d) = 4$; and we need to update the labels of the following vertices not in S: $L(f) = \min(8, 4+7) = 8$, $L(g) = \min(7, 4+2) = 6$ (the other labels remain unchanged).

The shortest path from a to d has weight 4 and consists of the sequence of edges $\{a, c\}, \{c, d\}$.

5. The smallest label value for vertices not in S is obtained for g. Thus, add g to S and check which of the vertices not in S that are connected via an edge to g (namely, vertices e, i, and j) may get lower label values. As a result we find: $S = \{a, b, c, d, g\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$, $L(d) = 4$, $L(g) = 6$; and we need to update the labels of the following vertices not in S: $L(e) = \min(10, 6+2) = 8$, $L(i) = \min(\infty, 6+9) = 15$, $L(j) = \min(\infty, 6+7) = 13$ (the other labels remain unchanged).

   The shortest path from a to g has weight 6 and consists of the sequence of edges $\{a, c\}, \{c, d\}, \{d, g\}$.

6. The smallest label value for vertices not in S is obtained for h. Thus, add h to S and check which of the vertices not in S that are connected via an edge to h (namely, vertices f and k) may get lower label values. As a result we find: $S = \{a, b, c, d, g, h\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$, $L(d) = 4$, $L(g) = 6$, $L(h) = 7$; and we need to update the labels of the following vertices not in S: $L(f) = \min(8, 7+3) = 8$, $L(k) = \min(\infty, 7+6) = 13$ (the other labels remain unchanged).

   The shortest path from a to h has weight 7 and consists of the sequence of edges $\{a, c\}, \{c, h\}$.

7. A smallest label value for vertices not in S is obtained for e. Thus, add e to S and check which of the vertices not in S that are connected via an edge to e (namely, vertex j) may get lower label value. As a result we find: $S = \{a, b, c, d, e, g, h\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$, $L(d) = 4$, $L(e) = 8$, $L(g) = 6$, $L(h) = 7$; and we need to update the label of the following vertex not in S: $L(j) = \min(13, 8+3) = 11$ (the other labels remain unchanged).

   The shortest path from a to e has weight 8 and consists of the sequence of edges $\{a, c\}, \{c, d\}, \{d, g\}, \{g, e\}$.

8. The smallest label value for vertices not in S is obtained for f. Thus, add f to S and check which of the vertices not in S that are connected via an edge to f (namely, vertices i and k) may get lower label values. As a result we find: $S = \{a, b, c, d, e, f, g, h\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$, $L(d) = 4$, $L(e) = 8$, $L(f) = 8$, $L(g) = 6$, $L(h) = 7$; and we need to update the labels of the following vertices not in S: $L(i) = \min(15, 8+7) = 15$, $L(k) = \min(13, 8+4) = 12$ ($L(i)$ does not change, but we find that there are multiple ways in which vertex i can be reached at cost 15; the other labels remain unchanged).

   The shortest path from a to f has weight 8 and consists of the sequence of edges $\{a, c\}, \{c, f\}$.

9. The smallest label value for vertices not in S is obtained for j. Thus, add j to S and check which of the vertices not in S that are connected via an edge to j (namely, vertices i and l) may get lower label values. As a result we find: $S = \{a, b, c, d, e, f, g, h, j\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$, $L(d) = 4$, $L(e) = 8$, $L(f) = 8$, $L(g) = 6$, $L(h) = 7$, $L(j) = 11$; and we need to update the labels of the following vertices not in S: $L(i) = \min(15, 11 + 3) = 14$, $L(l) = \min(\infty, 11 + 6) = 17$ (the other labels remain unchanged).

   The shortest path from a to j has weight 11 and consists of the sequence of edges $\{a, c\}, \{c, d\}, \{d, g\}, \{g, e\}, \{e, j\}$.

10. The smallest label value for vertices not in S is obtained for k. Thus, add k to S and check which of the vertices not in S that are connected via an edge to k (namely, vertices i and l) may get lower label values. As a result we find: $S = \{a, b, c, d, e, f, g, h, j, k\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$, $L(d) = 4$, $L(e) = 8$, $L(f) = 8$, $L(g) = 6$, $L(h) = 7$, $L(j) = 11$, $L(k) = 12$; and we need to update the labels of the following vertices not in S: $L(i) = \min(14, 12+2) = 14$, $L(l) = \min(17, 12+5) = 17$ (no changes are made, but we find there are multiple ways vertex i can be reached at cost 14, or l at cost 17; the other labels remain unchanged as well).

    The shortest path from a to k has weight 12 and consists of the sequence of edges $\{a, c\}, \{c, f\}, \{f, k\}$.

11. The smallest label value for vertices not in S is obtained for i. Thus, add i to S and check which of the vertices not in S that are connected via an edge to i (namely, vertex l) may get lower label value. As a result we find: $S = \{a, b, c, d, e, f, g, h, i, j, k\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$, $L(d) = 4$, $L(e) = 8$, $L(f) = 8$, $L(g) = 6$, $L(h) = 7$, $L(i) = 14$, $L(j) = 11$, $L(k) = 12$; and we need to update the label of the following vertex not in S: $L(l) = \min(17, 14 + 2) = 16$ (the other labels remain unchanged).

    The shortest path from a to i has weight 14. A weight 14 path from a to i consists of the sequence of edges $\{a, c\}, \{c, d\}, \{d, g\}, \{g, e\}, \{e, j\}, \{j, i\}$.

12. The smallest (and only) label value for vertices not in S is obtained for l. Thus, add l to S and note that there are no other vertices that need to be checked if they may get lower label values. As the final result we find: $S = \{a, b, c, d, e, f, g, h, i, j, k, l\}$ with shortest path values $L(a) = 0$, $L(b) = 4$, $L(c) = 3$, $L(d) = 4$, $L(e) = 8$, $L(f) = 8$, $L(g) = 6$, $L(h) = 7$, $L(i) = 14$, $L(j) = 11$, $L(k) = 12$, and $L(l) = 16$.

    The shortest path from a to l has weight 16. A weight 16 path from a to l consists of the sequence of edges

    $$\{a, c\}, \{c, d\}, \{d, g\}, \{g, e\}, \{e, j\}, \{j, i\}, \{i, l\}.$$

    (Because there were multiple shortest paths of weight 14 from a to i and because the above shortest path from a to l passes through i, the above shortest path is not the unique shortest path from a to l:

    $$\{a, c\}, \{c, f\}, \{f, k\}, \{k, i\}, \{i, l\}$$

has weight 16 too.)

12
For each subquestion below tick only the correct circle. In each subquestion G refers to the weighted graph below.

1. When Prim's algorithm is used to find a minimal spanning tree for G, then the edge $\{f, k\}$

   ◯ must belong to the resulting minimal spanning tree;

   ✓ may belong to the resulting minimal spanning tree;

   ◯ does not belong to the resulting minimal spanning tree.

   Prim's algorithm starts with an edge of smallest weight, and keeps adding the smallest weight edge that is connected to the already selected edges while avoiding creation of a circuit. Thus, the edges of G may be selected in the following order:

   (a) $\{c, d\}$
   (b) $\{d, g\}$
   (c) $\{g, e\}$
   (d) $\{a, c\}$
   (e) $\{b, g\}$
   (f) $\{e, j\}$
   (g) $\{i, j\}$
   (h) $\{i, k\}$
   (i) $\{i, l\}$
   (j) either $\{c, h\}$ or $\{f, k\}$
   (k) $\{f, h\}$

   It follows that the second circle must be ticked.

2. When Kruskal's algorithm is used to find a minimal spanning tree for G, then the edge $\{a, b\}$

   ◯ must belong to the resulting minimal spanning tree;

   ◯ may belong to the resulting minimal spanning tree;

   ✓ does not belong to the resulting minimal spanning tree.

   Kruskal's algorithm starts with an edge of smallest weight, and keeps adding any smallest weight edge while avoiding creation of a circuit. Thus, it first selects the single edge of weight one, then all four edges of weight two, and next the five edges of weight three, since these are the minimal weight choices and they do not form a circuit. Yet one more edge needs to be selected to make the result connected. This cannot be the edge $\{a, b\}$ because it would form a circuit. Thus, the third circle must be ticked.

3. The underlying unweighted graph of G

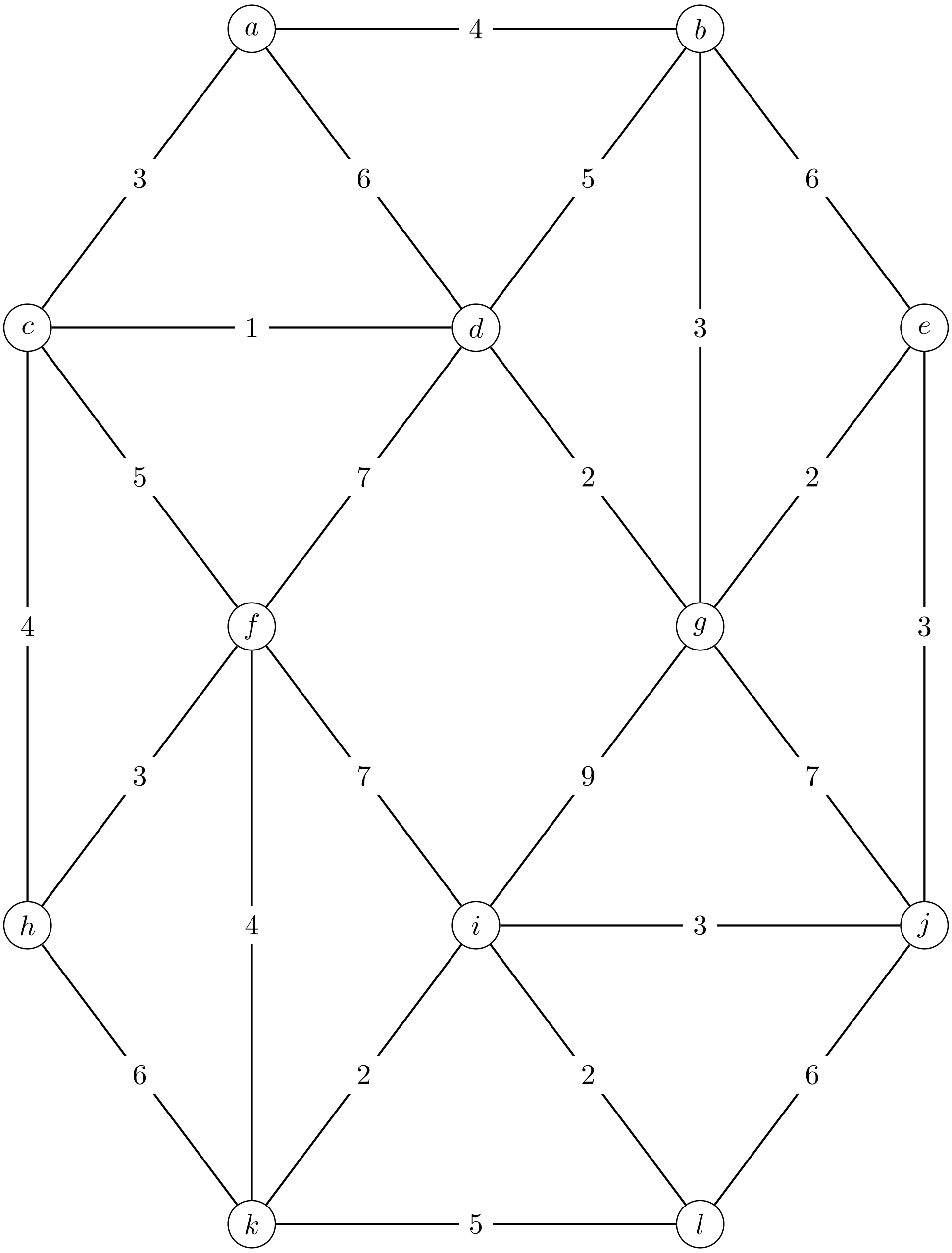
- ◯ allows an Euler circuit;
- ✓ allows an Euler circuit after removal of five appropriately chosen edges;
- ◯ allows an Euler path;
- ◯ allows an Euler path after removal of two appropriately chosen edges.

Removal of edges $\{a, d\}$, $\{e, g\}$, $\{f, i\}$, $\{h, k\}$, and $\{k, l\}$ results in a graph with only even vertex degrees. According to Theorem 1 on page 636 that graph has an Euler circuit. Thus, the second circle must be ticked.

4. The underlying unweighted graph of G

- ✓ allows a Hamilton circuit and it allows a Hamilton path from vertex b to vertex h;
- ◯ allows a Hamilton circuit but it does not allow a Hamilton path from vertex b to vertex h;
- ◯ allows a Hamilton path from vertex b to vertex h but it does not allow a Hamilton circuit;
- ◯ allows neither a Hamilton circuit nor a Hamilton path from vertex b to vertex h.

A Hamilton circuit is given by $a, c, f, h, k, i, l, j, g, e, b, d, a$, and a Hamilton path from b to h by $b, a, d, g, e, j, i, l, k, f, c, h$. Thus the first circle must be ticked.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