宽柔中學

(活页纸)

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10

3/4

Miscellaneous Examples 4

Find the coefficients of x^8 and x^{12} in the expansion of $(1-2x)^4(x+x^3)^7$.

For $(1-2x)^4$

$T_{r+1} = {}_4C_r (1)^{4-r} (-2x)^r$

$= {}_4C_r (1)^{4-r} (-2)^r x^r$

For $(x+x^3)^7$

$T_{k+1} = {}_7C_k (x)^{7-k} (x^3)^k$

$= {}_7C_k (x)^{7-k+3k}$

$= {}_7C_k x^{7+2k}$

$\left.\begin{matrix} x^r \\ x^{7+2k} \end{matrix}\right\} x^{r+7+2k} = x^8$

$r+7+2k = 8$

$r+2k = 1$

$r+2k = 1$

r	k	the coefficient of x^8	
1	0	$[{}_4C_1 (1)^{4-1} (-2)^1][{}_7C_0]$	$= -8$

$\therefore x^8 = -8$

$\left.\begin{matrix} x^r \\ x^{7+2k} \end{matrix}\right\} x^{r+7+2k} = x^{12}$

$r+2k = 5$

$r+2k = 5$

r	k	the coefficient of x^{12}	
1	2	$[{}_4C_1 (1)^{4-1} (-2)^1][{}_7C_2]$	$= -168$
3	1	$[{}_4C_3 (1)^{4-3} (-2)^3][{}_7C_1]$	$= -224$
~~5~~	~~0~~	$\cancel{[{}_4C_5 (1)^{4-5} (-2)^5][{}_7C_5]}$	-392

$\therefore x^{12} = -392$

rove that the $(n+1)$th term of a G.P., first term a and third term b is equal to the $(2n+1)$th term of a G.P., first term a and fifth term b.

$a_1 = a$

$a_3 = ar^2 = b$

$r^2 = \frac{b}{a}$

$r = \pm(\frac{b}{a})^{\frac{1}{2}}$

$a_n = ar^{n-1}$

$\therefore a_{n+1} = ar^{n+\cancel{1-1}}$

$= a(\pm\frac{b}{a})^{\frac{n}{2}}$

$a_1 = a$

$a_5 = ar^4 = b$

$r^4 = \frac{b}{a}$

$r = \pm(\frac{b}{a})^{\frac{1}{4}}$

$a_n = ar^{n-1}$

$\therefore a_{2n+1} = ar^{2n+\cancel{1-1}}$

$= ar^{2n}$

$= a(\pm\frac{b}{a})^{\frac{1}{4}\times 2n}$

$= a(\pm\frac{b}{a})^{\frac{n}{2}}$

$\therefore$ prove that the $(n+1)$th term is equal to the $(2n+1)$th term.

Find the term independent of a in $(\underset{a}{a} - \underset{b}{\frac{1}{a^2}})^{12 \to n}$.

$T_{r+1} = {}_nC_r\ a^{n-r}\ b^r$

$= {}_{12}C_r\ a^{12-r}\ (-\frac{1}{a^2})^r$

$= {}_{12}C_r\ a^{12-r}\ (-1)^r\ a^{-2r}$

$= {}_{12}C_r\ (-1)^r\ a^{12-r-2r}$

$\therefore a^{12-3r} = a^0$

$12-3r = 0$

$-3r = -12$

$r = 4$

$\therefore$ the term independent of $a = {}_{12}C_4\ (-1)^4$

$= 495$ ※

Sum to n terms and to infinity, the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \dots$

the nth term $= (3n-2)(\frac{1}{5})^{n-1}$

Let $1\ S_n = 1 + 4(\frac{1}{5}) + 7(\frac{1}{5})^2 + 10(\frac{1}{5})^3 + \dots + (3n-5)(\frac{1}{5})^{n-2} + (3n-2)(\frac{1}{5})^{n-1} + 0$

$\frac{1}{5} S_n = \frac{1}{5} + 4(\frac{1}{5})^2 + 7(\frac{1}{5})^3 + \dots + (3n-5)(\frac{1}{5})^{n-1} + (3n-2)(\frac{1}{5})^n$

$(1-\frac{1}{5}) S_n = 1 + 3(\frac{1}{5}) + 3(\frac{1}{5})^2 + 3(\frac{1}{5})^3 + \dots + 3(\frac{1}{5})^{n-1} - (3n-2)(\frac{1}{5})^n$

$= 1 + 3[\frac{1}{5} + (\frac{1}{5})^2 + (\frac{1}{5})^3 + \dots + (\frac{1}{5})^{n-1}] - (3n-2)(\frac{1}{5})^n$

$= 1 + 3\left[\frac{\frac{1}{5}(1-(\frac{1}{5})^{n-1})}{1-\frac{1}{5}}\right] - (3n-2)(\frac{1}{5})^n$

$\frac{4}{5} S_n = 1 + \frac{15}{4}[\frac{1}{5}(1-(\frac{1}{5})^{n-1})] - (3n-2)(\frac{1}{5})^n$

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（活页纸）

班 级 CLASS: 教 师 TEACHER:

姓 名 NAME: ( ) 日 期 DATE:

$\frac{4}{5}S_n = 1 + \frac{3}{4} - \frac{3}{4}(\frac{1}{5})^{n-1} - (3n-2)(\frac{1}{5})^n$

$\frac{4}{5}S_n = \frac{7}{4} - \frac{3}{4}(\frac{1}{5})^{n-1} - (3n-2)(\frac{1}{5})^n$

$S_n = \frac{35}{16} - \frac{5}{4}\left[\frac{3}{4(5)^{n-1}} + \frac{3n-2}{5^n}\right]$

$= \frac{35}{16} - \frac{5}{4}\left[\frac{3(5) + 4(3n-2)}{4(5)^n}\right]$

$= \frac{35}{16} - \frac{5}{4}\left[\frac{15 + 12n - 8}{4(5)^n}\right]$

$= \frac{35}{16} - \frac{\cancel{5}}{4}\left[\frac{12n+7}{4(5)^{\cancel{n}\,n-1}}\right]$

$= \frac{35}{16} - \frac{1}{16 \cdot 5^{n-1}}(12n+7)$ ※

the sum to infinity, $n \to \infty$

$S_\infty = \frac{35}{16} - \frac{1}{16 \cdot 5^{n-1}}(12n+7)$

$= \frac{35}{16}$ ※

5\. Sum to n terms, $1 \cdot 5 + 3 \cdot 7 + 5 \cdot 9 + \ldots$

$a_n = a_1 + (n-1)d$ $\quad a_n = a_1 + (n-1)d$

$= 1 + (n-1)(2)$ $\quad = 5 + (n-1)(2)$

$= 1 + 2n - 2$ $\quad = 5 + 2n - 2$

$= 2n - 1$ $\quad = 2n + 3$

$a_n = (2n-1)(2n+3)$

the general term : $U_r = (2r-1)(2r+3)$

$= 4r^2 + 6r - 2r - 3$

$= 4r^2 + 4r - 3$

$\sum_1^n U_r = \sum_1^n (4r^2 + 4r - 3)$

$= 4\sum_1^n r^2 + 4\sum_1^n r - \sum_1^n 3$

$= \cancel{4}[\frac{1}{6}n(n+1)(2n+1)] + \cancel{4}[\frac{1}{2}n(n+1)] - 3n$

$= \frac{2}{3}n(2n^2+3n+1) + 2n^2 + 2n - 3n$

$= \frac{4}{3}n^3 + 2n^2 + \frac{2}{3}n + 2n^2 + 2n - 3n$

$= \frac{4}{3}n^3 + 4n^2 - \frac{1}{3}n = \frac{1}{3}n(4n^2 + 12n - 1)$ ※

$$nC_r = \frac{n!}{(n-r)!\,r!}$$

6. If the coefficients of the first three terms in the expansion of $(1+\frac{x}{2})^n$ are in A.P., find the 4th term.

The coefficient of 1st term,

$nC_0 \ (1)^n \ (\frac{1}{2})^0$

$= 1$

The coefficient of 2nd term,

$nC_1 \ (1)^{n-1} \ (\frac{1}{2})^1$

$= \frac{1}{2}n$

The coefficient of 3rd term,

$nC_2 \ (1)^{n-2} \ (\frac{1}{2})^2$

$\frac{n!}{(n-2)!\,2!} \ (\frac{1}{2})^2 = \frac{n(n-1)}{2} \ (\frac{1}{4})$

$= \frac{n^2-n}{8}$

$d = \frac{1}{2}n - 1$

$a_3 = 1 + (3-1)(\frac{1}{2}n - 1)$

$= 1 + (\frac{3}{2}n - 3 - \frac{1}{2}n + 1)$

$= 1 + (n-2)$

$= n-1$

$\therefore \ n-1 = \frac{n^2-n}{8}$

$n^2 - n = 8n - 8$

$n^2 - 9n + 8 = 0$

$(n-8)(n-1) = 0$

$n = 8$ or $n = 1$ (rejected)

$n \quad -8$

$n \quad -1$

$-n - 8n = -9n$

$\therefore$ the 4th term $= {}_8C_3 \ (1)^{8-3} \ (\frac{1}{2})^3 \ x^3$

$= 7x^3$ *

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Find the first three terms in the expansions of $(1+x)^{10}$ and $(1+2x)^7$. If $x=\frac{4}{3}$ show that the fifth term of the first is equal to the fourth term of the second and if x has this value, find the numerically greatest term in the expansion of $(1+2x)^7$.

For $(1+x)^{10} = 1 + {}_{10}C_1 x + {}_{10}C_2 x^2 + {}_{10}C_3 x^3 + {}_{10}C_4 x^4 + \ldots$

$= 1 + 10x + 45x^2 + 120x^3 + 210x^4 + \ldots$

For $(1+2x)^7 = 1 + {}_7C_1\, 2x + {}_7C_2 (2x)^2 + {}_7C_3 (2x)^3 + {}_7C_4 (2x)^4$

$= 1 + 14x + 84x^2 + 280x^3 + 560x^4 + \ldots$

$\therefore (1+x)^{10} = 1 + 10x + 45x^2 + \ldots$

$(1+2x)^7 = 1 + 14x + 84x^2 + \ldots$ ※

the fifth term of the first

$210\left(\frac{4}{3}\right)^4 = \frac{17920}{27}$

the fourth term of the second

$280\left(\frac{4}{3}\right)^3 = \frac{17920}{27}$

$n=7,\ a=1,\ b=2,\ x=\frac{4}{3}$

$r \le \frac{(n+1)\,bx}{1+bx}$

$r \le \frac{(7+1)(2)\left(\frac{4}{3}\right)}{1+(2)\left(\frac{4}{3}\right)}$

$r \le 5.82$

$\therefore r = 5$

The greatest term is the 6th term ※

The 6^{th} term :

$= {}_7C_5\,(1)^{7-5}\left[2\left(\frac{4}{3}\right)\right]^5$

$= \frac{7 \cdot 2^{15}}{3^4}$ ※

1.1 P42

The first two terms of a G.P. are a and b ($b < a$). If the sum of the first n terms is equal to the sum to infinity of the remaining terms, prove $a^n = 2b^n$.

$r = \frac{b}{a}$

$S_n = \frac{a(1-r^n)}{1-r}$ $\qquad$ $S_\infty = \frac{a}{1-r}$

$= \frac{a(1-(\frac{b}{a})^n)}{1-\frac{b}{a}}$ $\qquad$ $= \frac{a}{1-\frac{b}{a}}$

$S_n = S_\infty - S_n$

$2S_n = S_\infty$

$2\left[\frac{\cancel{a}(1-(\frac{b}{a})^n)}{\cancel{1-\frac{b}{a}}}\right] = \frac{\cancel{a}}{\cancel{1-\frac{b}{a}}}$

$2[1-(\frac{b}{a})^n] = 1$

$2 - \frac{2b^n}{a^n} = 1$

$2 - 1 = \frac{2b^n}{a^n}$

$1 = \frac{2b^n}{a^n}$

$a^n = 2b^n$ ※ (proved)

Sum to n terms, $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \dots$

$a_n = a_1 + (n-1)d$ $\qquad$ $a_n = a_1 + (n-1)d$

$= 1 + (n-1)(3)$ $\qquad$ $= 4 + (n-1)(3)$

$= 1 + 3n - 3$ $\qquad$ $= 4 + 3n - 3$

$= 3n - 2$ $\qquad$ $= 3n + 1$

The general term $= \frac{1}{(3r-2)(3r+1)}$

$\frac{1}{(3r-2)(3r+1)} \equiv \frac{A}{3r-2} + \frac{B}{3r+1}$

$1 = A(3r+1) + B(3r-2)$

when $r = -\frac{1}{3}$, $-3B = 1 \Rightarrow B = -\frac{1}{3}$

when $r = \frac{2}{3}$, $3A = 1 \Rightarrow A = \frac{1}{3}$

$\therefore \frac{1}{(3r-2)(3r+1)} = \frac{1}{3(3r-2)} - \frac{1}{3(3r+1)}$

$\sum_{1}^{n}\left[\frac{1}{3(3r-2)} - \frac{1}{3(3r+1)}\right] = \frac{1}{3}\sum_{1}^{n}\left[\frac{1}{3r-2} - \frac{1}{3r+1}\right]$

$= \frac{1}{3}(1 - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{10}} + \dots + \cancel{\frac{1}{3n-2}} - \frac{1}{3n+1})$

$= \frac{1}{3}(1 - \frac{1}{3n+1})$

$= \frac{1}{\cancel{3}}(\frac{\cancel{3}n}{3n+1})$

$= \frac{n}{3n+1}$ ※

10. Find the greatest coefficient in the expansion of $(3+7x)^{10}$.

$(3+7x)^{10} = 3^{10}\left(1+\frac{7}{3}x\right)^{10}$

$r \leq \frac{(n+1)\ b}{1+b}$

$r \leq \frac{(10+1)\left(\frac{7}{3}\right)}{1+\frac{7}{3}}$

$r \leq 7.7 \quad \therefore r = 7$

The greatest coefficient = the 8th term.

$\therefore$ the greatest coefficient is

$T_8 = {}_{10}C_7 (1)^{10-7} \left(\frac{7}{3}\right)^7 \cdot 3^{10}$

$= 120 \cdot \left(\frac{7}{3}\right)^7 \cdot 3^{10}$ ※

11. Sum to n terms each of the series:

(i) $1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \ldots$

$a_n = a_1 + (n-1)d$
$= 1 + (n-1)(1)$
$= 1 + n - 1$
$= n$

$a_n = a_1 + (n-1)d$
$= 2 + (n-1)(1)$
$= 2 + n - 1$
$= n+1$

the general term $= r^2(r+1)$

$\sum_1^n [r^2(r+1)] = \sum_1^n (r^3 + r^2)$
$= \sum_1^n r^3 + \sum_1^n r^2$
$= \left[\frac{1}{2}n(n+1)\right]^2 + \left[\frac{1}{6}n(n+1)(2n+1)\right]$
$= \frac{1}{12}n(n+1)\left[3(n)(n+1) + 2(2n+1)\right]$
$= \frac{1}{12}n(n+1)(3n^2 + 3n + 4n + 2)$
$= \frac{1}{12}n(n+1)(3n^2 + 7n + 2)$
$= \frac{1}{12}n(n+1)(n+2)(3n+1)$ ※

3n 1
n 2
6n + n = 7n

(ii) $1 + 3x + 5x^2 + 7x^3 + \ldots$

$a_n = a_1 + (n-1)(d)$
$= 1 + (n-1)(2)$
$= 1 + 2n - 2$
$= 2n - 1$

$a_n = a_1 r^{n-1}$
$= (1)(x)^{n-1}$
$= x^{n-1}$

the general term $= (2n-1)x^{n-1}$

Let $S_n = 1 + 3x + 5x^2 + 7x^3 + \ldots + (2n-3)x^{n-2} + (2n-1)x^{n-1} + 0$

$x S_n = x + 3x^2 + 5x^3 + \ldots + (2n-3)x^{n-1} + (2n-1)x^n$

$(1-x) S_n = 1 + 2x + 2x^2 + 2x^3 + \ldots \ldots \ldots + 2x^{n-1} - (2n-1)x^n$

$= 1 + 2(x + x^2 + x^3 + \ldots + x^{n-1}) - (2n-1)x^n$

$= 1 + 2\left[\frac{x(1-x^{n-1})}{1-x}\right] - (2n-1)x^n$

$S_n = \frac{1}{1-x} + \frac{2[x(1-x^{n-1})]}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$

$= \frac{1-x + 2x(1-x^{n-1}) - [(2n-1)x^n(1-x)]}{(1-x)^2}$

$= \frac{1-x+2x-2x^n-[(2nx^n-x^n)(1-x)]}{(1-x)^2}$

$= \frac{1-x+2x-2x^n-(2nx^n-2nx^{n+1}-x^n+x^{n+1})}{(1-x)^2}$

$= \frac{1-x+2x-2x^n-2nx^n+2nx^{n+1}+x^n-x^{n+1}}{(1-x)^2}$

$= \frac{1+x-x^n(2n+1)+x^{n+1}(2n-1)}{(1-x)^2}$ ※

12. Show that the nth term of the series

$3 + 4x + 6x^2 + 10x^3 + 18x^4 + 34x^5 + \ldots$

is $2x^{n-1} + (2x)^{n-1}$ and hence find the sum to n terms.

$2^0 + 2 = 3$ $2^1 + 2 = 4$ $2^2 + 2 = 6$ $2^3 + 2 = 10$

x^0 x^1 x^2 x^3

$\therefore$ n term $= (2^{n-1} + 2)x^{n-1}$

$= (2x)^{r-1} + (2 \cdot x^{n-1})$

$U_r = (2x)^{r-1} + (2 \cdot x^{r-1})$

$\sum_1^n U_r = \sum_1^n [(2x)^{r-1} + (2 \cdot x^{r-1})]$

$= \sum_1^n (2x)^{r-1} + 2\sum_1^n x^{r-1}$

$= \frac{1-(2x)^n}{1-2x} + 2 \cdot \frac{1-x^n}{1-x}$ ※

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班级 CLASS: 教师 TEACHER:

姓名 NAME: ( ) 日期 DATE:

13. Prove by induction.

$\frac{1}{3\cdot4\cdot5}+\frac{2}{4\cdot5\cdot6}+\frac{3}{5\cdot6\cdot7}+\ldots$ n terms $=\frac{1}{6}-\frac{1}{n+3}+\frac{2}{(n+3)(n+4)}$

$a_n = a_1+(n-1)(d)$ $\quad$ $a_n = a_1+(n-1)(d)$ $\quad$ $a_n = a_1+(n-1)(d)$ $\quad$ $a_n = a_1+(n-1)(d)$

$= 1+(n-1)(1)$ $\quad$ $= 3+(n-1)(1)$ $\quad$ $= 4+(n-1)(1)$ $\quad$ $= 5+(n-1)(1)$

$= \cancel{1}+n\cancel{-1}$ $\quad$ $= 3+n-1$ $\quad$ $= 4+n-1$ $\quad$ $= 5+n-1$

$= n$ $\quad$ $= n+2$ $\quad$ $= n+3$ $\quad$ $= n+4$

n term $=\frac{n}{(n+2)(n+3)(n+4)}$

$\frac{1}{3\cdot4\cdot5}+\frac{2}{4\cdot5\cdot6}+\frac{3}{5\cdot6\cdot7}+\ldots+\frac{n}{(n+2)(n+3)(n+4)}=\frac{1}{6}-\frac{1}{n+3}+\frac{2}{(n+3)(n+4)}$

Step 1 : $n=1$, L.H.S $=\frac{1}{3\cdot4\cdot5}=\frac{1}{60}$

R.H.S $=\frac{1}{6}-\frac{1}{4}+\frac{2}{4\cdot5}$

$=\frac{1}{60}$

$\therefore$ L.H.S = R.H.S

Step 2 : Assuming $n=k$, the statement is true.

$\frac{1}{3\cdot4\cdot5}+\frac{1}{4\cdot5\cdot6}+\frac{1}{5\cdot6\cdot7}+\ldots+\frac{k}{(k+2)(k+3)(k+4)}=\frac{1}{6}-\frac{1}{k+3}+\frac{2}{(k+3)(k+4)}$

Let $n=k+1$,

$\frac{1}{3\cdot4\cdot5}+\frac{1}{4\cdot5\cdot6}+\frac{1}{5\cdot6\cdot7}+\ldots+\frac{k}{(k+2)(k+3)(k+4)}+\frac{k+1}{(k+3)(k+4)(k+5)}$

$=\frac{1}{6}-\frac{1}{k+3}+\frac{2}{(k+3)(k+4)}+\frac{k+1}{(k+3)(k+4)(k+5)}$

$=\frac{1}{6}-\frac{1}{k+3}+\frac{1}{(k+3)(k+4)}+\frac{1}{(k+3)(k+4)}+\frac{k+1}{(k+3)(k+4)(k+5)}$

$=\frac{1}{6}-\frac{k+4-1}{(k+3)(k+4)}+\frac{k+5+k+1}{(k+3)(k+4)(k+5)}$

$=\frac{1}{6}-\frac{\cancel{k+3}}{(\cancel{k+3})(k+4)}+\frac{2k+6}{(k+3)(k+4)(k+5)}$

$=\frac{1}{6}-\frac{1}{k+4}+\frac{2(\cancel{k+3})}{(\cancel{k+3})(k+4)(k+5)}$

$=\frac{1}{6}-\frac{1}{k+4}+\frac{2}{(k+4)(k+5)}$

$=\frac{1}{6}-\frac{1}{k+1+3}+\frac{2}{(k+1+3)(k+1+4)}$ *

$\therefore$ The result is true for all $n\in N$.

16. Find the numerically greatest term in $(3-2x)^{11}$ when $x = \frac{3}{5}$.

$(3-2x)^{11} = 3^{11}(1 - \frac{2}{3}x)^{11}$

$r \leq \frac{(n+1)bx}{1+bx}$

$r \leq \frac{(11+1)(\frac{2}{3})(\frac{3}{5})}{1+(\frac{2}{3})(\frac{3}{5})}$

$r \leq 3.43 \quad \therefore r = 3$

The greatest term is 4th term. ※

$T_4 = {}_{11}C_3 (1)^{11-3}[(\frac{2}{3})(\frac{3}{5})]^3 \cdot 3^{11}$

$= 165 (\frac{2}{5})^3 \cdot 3^{11}$ ※

17. Prove that the sum of the series

$(2n-1) + 2(2n-3) + 3(2n-5) + \ldots n \text{ terms} = \frac{1}{6}n(n+1)(2n+1)$

$a_n = a_1 + (n-1)d$

$= 1 + (n-1)(1)$

$= \cancel{1} + n - \cancel{1}$

$= n$

$a_n = a_1 + (n-1)d$

$= 1 + (n-1)(2)$

$= 1 + 2n - 2$

$= 2n - 1$

$U_r = r[2n - (2r-1)]$

$\sum_1^n U_r = \sum_1^n r[2n-(2r-1)]$

$= 2\sum_1^n rn - 2\sum_1^n r^2 + \sum_1^n r$

$= 2n[\frac{1}{2}n(n+1)] - 2[\frac{1}{6}n(n+1)(2n+1)] + \frac{1}{2}n(n+1)$

$= \frac{2n}{2} n(n+1) - \cancel{2}[\frac{n(n+1)(2n+1)}{6 \; 3}] + \frac{n(n+1)}{2}$

$= n^2(n+1) \cdot 6 - \frac{n(n+1)(2n+1) \cdot 2}{3 \cdot 2} + \frac{n(n+1) \cdot 3}{2 \cdot 3}$

$= \frac{6n^2(n+1)}{6} - \frac{2n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{6}$

$= \frac{n(n+1)}{6}[6n - 2(2n+1) + 3]$

$= \frac{1}{6}n(n+1)(6n - 4n - 2 + 3)$

$= \frac{1}{6}n(n+1)(2n+1)$ ※ (proved)

18. Find an A.P. with first term unity such that the second, tenth and thirty-fourth terms form a geometric series.

$1 + 1+d + 1+9d + 1+33d$

$\frac{1+9d}{1+d} = \frac{1+33d}{1+9d}$

$(1+d)(1+33d) = (1+9d)(1+9d)$

$1+33d+d+33d^2 = 1+9d+9d+81d^2$

$1+34d+33d^2 = 1+18d+81d^2$

$1-1+34d-18d+33d^2-81d^2 = 0$

$-48d^2+16d = 0$

$48d^2-16d = 0$

$16d(3d-1) = 0$

$16d = 0$ or $3d-1 = 0$

$d = 0$ (rejected) $d = \frac{1}{3}$ ※

$\therefore$ common difference $= \frac{1}{3}$

19. Find the term independent of x in $\left(\frac{4x^2}{3} - \frac{3}{2x}\right)^9$.

$T_{r+1} = {}^9C_r \left(\frac{4x^2}{3}\right)^{9-r} \left(-\frac{3}{2x}\right)^r$

$= {}^9C_r \left(\frac{4}{3}\right)^{9-r} (x^2)^{9-r} (-1)^r \left(\frac{3}{2}\right)^r x^{-r}$

$= {}^9C_r \left(\frac{4}{3}\right)^{9-r} (-1)^r \left(\frac{3}{2}\right)^r x^{18-2r-r}$

$\therefore 18-2r-r = 0$

$18-3r = 0$

$3r = 18$

$r = 6$

$\therefore$ the term independent of x $= {}^9C_6 \left(\frac{4}{3}\right)^{9-6} (-1)^6 \left(\frac{3}{2}\right)^6$

$= 2268$ ※

宽柔中学

(活页纸)

班级 CLASS:

教师 TEACHER:

姓名 NAME: ( )

日期 DATE:

20. By squaring the result $1+2+3+4+\dots+n = \frac{n(n+1)}{2}$, find the sum of the products of the first n natural numbers taken two at a time and show that it is equal to half the difference between the sum of their cubes and the sum of their squares.

$1+2+3+4+\dots+n = \frac{n(n+1)}{2}$

Squaring both sides,

$(1+2+3+4+\dots+n)^2 = \left[\frac{n(n+1)}{2}\right]^2$

a) $1^2+2^2+3^2+4^2+\dots+n^2+2\sum\alpha\beta = \sum_1^n r^3$

$\therefore \sum_1^n r^2 + 2\sum\alpha\beta = \sum_1^n r^3$

$2\sum\alpha\beta = \sum_1^n r^3 - \sum_1^n r^2$

$\sum\alpha\beta = \frac{1}{2}\left\{\sum_1^n r^3 - \sum_1^n r^2\right\}$ *

Note: $(a+b)^2 = a^2+2ab+b^2$

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$

$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$

$(a+b+c+d+e)^2 = a^2+b^2+c^2+d^2+e^2+2ab+2ac+2ad+2ae+2bc+2bd+2be+2cd+2ce+2de$

⋮

etc.

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日 期 DATE:

23. If $S_n = 3 + 8a + 15a^2 + 24a^3 + \dots n(n+2)a^{n-1}$, obtain the value of $S_n(1-a)^2$ and deduce the value of S_n. If $a = \frac{1}{3}$, what is the limit of S_n as $n \to \infty$?

$$S_n = 3 + 8a + 15a^2 + 24a^3 + \dots + n(n+2)a^{n-1} \quad — ①$$

$$aS_n = 3a + 8a^2 + 15a^3 + 24a^4 + \dots + (n-1)(n+1)a^{n-1} + n(n+2)a^n \quad — ②$$

① - ② :

$$(1-a)S_n = 3 + 5a + 7a^2 + 9a^3 + \dots + (2n+1)a^{n-1} - n(n+2)a^n \quad — ③$$

③ × a :

$$a(1-a)S_n = 3a + 5a^2 + 7a^3 + 9a^4 + \dots + (2n-1)a^{n-1} + (2n+1)a^n - n(n+2)a^{n+1} \quad — ④$$

$$[(1-a) - a(1-a)]S_n = 3 + \overbrace{2a + 2a^2 + 2a^3 + \dots + 2a^{n-1}}^{r=a,\ G.P\ of\ (n-1)\ term} - (n^2+4n+1)a^n + (n^2+2n)a^{n+1}$$

$$[(1-a) - a(1-a)]S_n = 3 + \frac{2a(1-a^n)}{1-a} - (n^2+4n+1)a^n + (n^2+2n)a^{n+1}$$

$a - a + a^2$

$1 - 2a + a^2$

$(1-a)^2$

$$(1-a)^2 S_n = 3 + \frac{2a(1-a^n)}{1-a} - (n^2+4n+1)a^n + (n^2+2n)a^{n+1}$$

$$S_n = \frac{3}{(1-a)^2} + \frac{2a(1-a^n)}{(1-a)^3} - \frac{(n^2+4n+1)a^n + (n^2+2n)a^{n+1}}{(1-a)^2} \quad *$$

when $a = \frac{1}{3}$, $n \to \infty$

$$S_n = \frac{3}{(1-a)^2} + \frac{2a}{(1-a)^3}$$

$$= \frac{3}{[1-(\frac{1}{3})]^2} + \frac{2(\frac{1}{3})}{[(1-\frac{1}{3})]^3}$$

$$= \frac{3}{(\frac{2}{3})^2} + \frac{\frac{2}{3}}{(\frac{2}{3})^3}$$

$$= \frac{27}{4} + \frac{9}{4}$$

$$= 9 \quad *$$

24. Show that in the expansion of $(1+x)^n$ where x is positive, the terms will increase until a term containing x^r is reached where r is the least integer $\geq \frac{nx-1}{x+1}$.
Calculate the greatest term in the expansion of $(1+2x)^{12}$ where $x = \frac{1}{4}$.

$$T_{r+1} \geq T_{r+2}$$

$$nC_r \cdot (1)^{n-r}(x)^2 \geq nC_{r+1} \cdot (1)^{n-r+1} \cdot (x)^{r+1}$$

$$nC_r \cdot x^r \geq nC_{r+1} \cdot x^{r+1}$$

$$\frac{n!}{(n-r)!\,r!}\,x^r \geq \frac{n!}{(n-r-1)!\,(r+1)!}\,x^{r+1}$$

$$\frac{\frac{\cancel{n!}}{(n-r-1)!\,(r+1)!} \cdot x^{\cancel{r+1}}}{\frac{\cancel{n!}}{(n-r)!\,r!} \cdot \cancel{x^r}} \geq 1$$

$$\frac{(n-r)!\;r!}{(n-r-1)!\;(r+1)!} \cdot x \leq 1$$

$$\frac{n-r}{r+1} \cdot x \leq 1$$

$$nx - xr \leq r+1$$

$$nx - 1 \leq r + xr$$

$$nx - 1 \leq r(1+x)$$

$$r \geq \frac{nx-1}{1+x}$$

*

$(1+2x)^{12}$, $x = \frac{1}{4}$

$$r \leq \frac{(12+1)(2)(\frac{1}{4})}{1+(2)(\frac{1}{4})}$$

$$r \leq \frac{\frac{13}{2}}{\frac{3}{2}}$$

$$r \leq \frac{13}{3}$$

$$r \leq 4\tfrac{1}{3}$$

take $r = 4$

The greatest term

$$T_{4+1} = 12C_4 \cdot (1)^{12-4} \cdot [(2)(\tfrac{1}{4})]^4$$

$$= 12C_4 \cdot (\tfrac{1}{2})^4$$

$$= \frac{495}{16}$$

*