HARMONIOUS PROPORTIONS

IN A PIANOFORTE - THE C.HA.S.® TEMPERAMENT

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English version: Liz Poore

http://matematica.unibocconi.it/articoli/relazioni-armoniche-un-pianoforte

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THE COMPROMISE

In 1691 the German organist **Andreas Werckmeister** discovered an ingenious way of tuning instruments, the closest ever achieved to an *equal temperament* [1], that is to say, to a tone system where the distance between semitones (two successive notes in the chromatic scale) is constant. An "exact" equal tuning system was inconceivable before the existence of electronics, given that the exact distance between semitones [2] is

$$\sqrt[12]{2} = 1.059463094\ldots$$

an irrational algebraic number, not a number that can be rendered geometrically. Werckmeister's acoustic compromise, which he named *good temperament*, was based on the combining of two other well-known and long-established systems. Using seven *Pythagorean* fifths (based on a 3:2 ratio) and five *mesotonic* fifths (more diminishing, built on the 5:4 ratio of a third) he managed to almost exactly complete the *cycle of 12 fifths* that "almost" exactly corresponded to 7 octaves :

$$\underbrace{\left(\sqrt[4]{5}\right)^5}_{\text{5 quinte mesotoniche}} \cdot \underbrace{\left(\frac{3}{2}\right)^7}_{\text{7 quinte pitagoriche}} = 5\sqrt[4]{5}\left(\frac{3}{2}\right)^7 \approx$$

$$\approx \underbrace{127,75 \approx 128}_{differenza=0.25} = \underbrace{2^7}_{\text{7 ottave}}$$

This approximation could not be achieved with only Pythagorean (or natural) fifths

$$\left(\frac{3}{2}\right)^{12} \approx \underbrace{129,75 > 128}_{differenza=-1.75} = 2^7$$

nor with only mesotonic fifths

$$\left(\sqrt[4]{5}\right)^{12} = 5^3 = \underbrace{125 < 128}_{differenza=3} = 2^7$$

Werckmeister's scale was extremely successful because of J. S. Bach's use of it in his "*Well-tempered Clavichord*" (1722 and 1744), for 24 preludes and 24 fugues in the 24 keys available (one for each note of the scale, in major and minor mode). This great change to music three centuries ago was an ingenious compromise between the musicians' need for "just", natural chords, and the mathematicians' need for "exact", irrational intervals.

FREQUENCIES OR BEATS?

The traditional method of tuning has been used for centuries, and is still widely used by those who find technological gadgetry, such as frequency metres and electronic tuners, unsatisfactory. It is based on beats; two notes separated by around twenty Hertz generate a sound that pulses in time, and that can be unpleasant to listen to or can be deliberately created (for example in organs) to achieve certain acoustic effects. Tuners have traditionally always aimed to reduce the beats resulting from the "unnatural" approximations of notes, as far as possible. But until now no one had ventured to investigate and understand this any further; instead the more common approach of frequency temperament was followed.

After more than thirty years of research and experimentation, **Alfredo Capurso**, a master tuner of international calibre, has put forward an interesting and revolutionary solution to this millennia-long tuning question; it is based on a few simple ideas. The first is to not insist on the octave ratio of 2:1, or Pythagorean equalness

octave = double frequency

(which means having to fit 12 semitones between two consecutive Dohs) and instead to radically alter practice and find notes that play "differently", even at a distance of a few Hz from their normal frequency.

The fact is that the tempered system mathematically resolves the problem of intervals (proportions between frequencies) in a scale, but it entails having to approximate all the intervals except for the octave. Tuners and those with greater harmonic-acoustic sensitivity have always had to accept the drawbacks of the unnatural approximation that, on the one hand, facilitated their work, but, on the other, did not do justice to the ear and to the full mosaic of all the 88 main frequencies of a pianoforte.

It seems that over time there has been an excessive concentration on the mathematization of music, resulting in the rather rigid fundamental theorem of Harmony, based on arithmetic theory:

Every interval R of tonal music may be expressed in one and only one way as a combination of octaves, fifths and major thirds:

$$R = 2^x\, 3^y\, 5^z \quad \text{con } x, y, z \in \text{¢}$$

Since the three base numbers are equally important, it seems pointless to ignore any of the three, or to try to build interval R without using octaves $\left(2^n\, 3^0\, 5^0\right)$, fifths $\left(2^{-n}\, 3^n\, 5^0\right)$ or major thirds $\left(2^{-2n}\, 3^0\, 5^n\right)$. Nevertheless, with equal temperament, the octave is the only "postulate" underlying the system, which finds compromise in the major third (5:4) and fifth (3:2), universally recognised as having greatest consonance. And here, after years of study, reflection and experimentation, is the light that Capurso saw: "The fact that a lower Doh is exactly half of the Doh an octave above is mere mathematical arbitrariness; the key to discovering the true proportion between one note and another is in the synchrony between sounds and their beats".

This idea has been put forward in physics and in particular in studies on psychoacoustics by **H. von Helmoltz**, for whom the consonance perceived between two sounds derives from the fact that the beats which they and their respective harmonics generate are "weak" in comparison to what happens in situations of dissonance (such as the "wolf fifth").

Capurso proposes a system which could be said to be the opposite of "*microtonal music*", that is the fragmentation of the octave into a variable number of tones. He stretches the octave to slightly over the Pythagorean 2:1 ratio, which he considers to have "no logical or practical basis".

As we know, a taut vibrating string producing a fundamental *Doh* of a frequency, let us say, equal to 1, at the same time produces many secondary notes (harmonics) that are whole multiples (2, 3, 4…) of that *Doh*, and of gradually decreasing intensity, so that at a certain point they are no longer audible. All the harmonics contribute towards determining the "form" of the sound, or its particular timbre.

Harmonic partial 2 (first octave), which has been in a privileged position up to now, unfairly takes space from partials 3, 4 (double octave) and 5. This is the next stage in the *harmonious system*: whereas the dichotomy between **frequency** proportions 2:1 and 3:2 does not allow the cycle of fifths to come to completion, a solution can be found in **beats frequencies**. As Capurso puts it: "A harmony Root can be found that recurs regardless of the dimensions of the generating sounds".

The resulting system, called Circular Harmonic System (C.Ha.S.®) by Capurso, has some extremely interesting characteristics. The harmony Root of the Chas System finds the precise beats proportion relative to partials 4 and 3.

HARMONIC PROPORTIONS

The Chas temperament, according to Capurso, expresses superlative harmoniousness, in terms of relationship between sounds and chords, between single notes and the whole set of the 88 keys of the pianoforte. It is a temperament that is no longer based on the numeric relationship between single notes, but on the relationship of any two notes to plurality, to the whole.

The limitation of the first equal temperament system is, as mentioned above, the result of two arbitrary choices in the first octave:

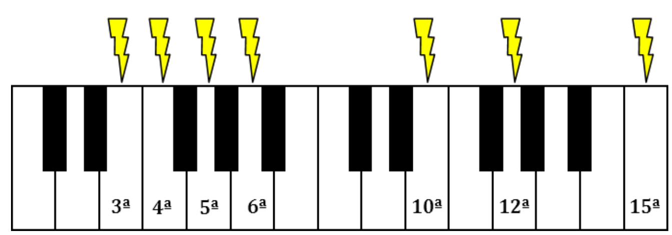
- the numeric base of 12 semitones;
- the fixed numeric relationship of semitone +12 (double the frequency of the first note +0).

The values of the first 13 sounds are generally repeated (in a sort of "cut and paste") for lower and higher octaves, with further "adjustments" that inevitably disrupt the proportions between the fundamentals and harmonics of the complete set of 88 notes. The solution here is to take a wider range, opening up the interval of reference to two octaves instead of one.

For Capurso it is not sufficient to establish a geometric ratio *k* (semitone) to obtain subsequent notes. Instead he uses "a System oriented towards pairs of sounds, so as to establish a multi-directional set where every semitone-sound gives the harmonic meaning and memory of every other sound, and where any interval (pair of notes) shows itself to be just and true". Thus the right proportions are not to be sought in the frequencies of the first octave only, but also in the beats expressed by pairs of notes with the right frequencies: they can express a kind of "restfulness" (consonance) or a variable "tension" (dissonance) created by harmonics.

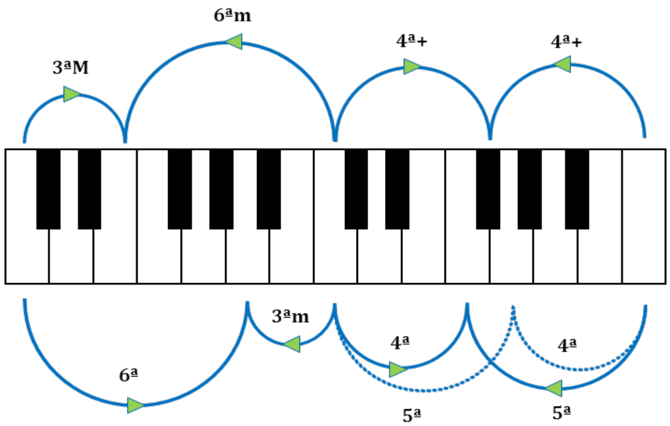
The "*tensor agents*" or key intervals that together with the octave express a particular tension are:

- the 3rd, 4th, 5th and 6th in the first octave;
- the 10th, 12th and 15th in the second octave.



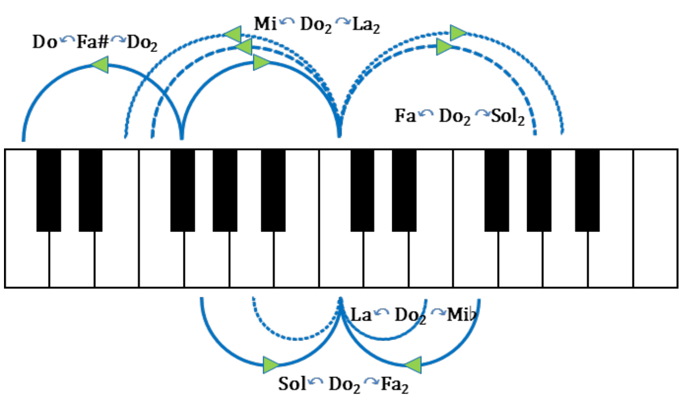
The usual harmonic symmetries between intervals are considered:

- major 3rd ↱ minor 6th and vice versa;
- 6th ↱ minor 3rd;
- 4th ↱ 5th and vice versa;
- augmented 4th, in the octave centre, and itself.



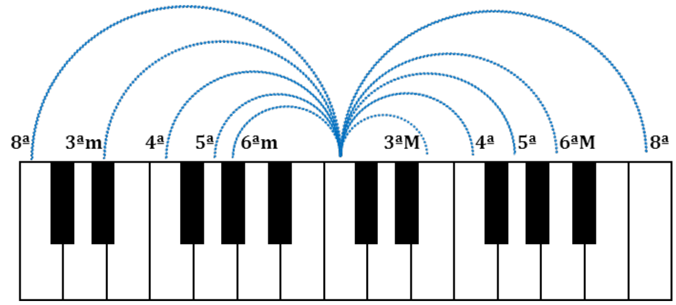
In the sequence Doh1 (semitone +0), Fa# (+6), Doh$_2$ (+12), Fa#$_2$ (+18), Doh$_3$ (+24) every note alternates in central position, thus creating five harmonic links (not just the octave) which in the +0∩+12 arc are:

- Do Fa# Do$_2$ (d5+ and P8);
- Mi Do$_2$ La$_2$ (M3, P8 and m6);
- Fa Do$_2$ Sol$_2$ (P4, P8 and P5);
- Sol Do$_2$ Fa$_2$ (P5,P8 and P4);
- La Do$_2$ Mi|$_2$ (M6, P8 and m3).



Thus it is clear that within two octaves each semitone has 5+5 harmonic links:

- to the right: M3 (major 3rd), P4, P5, M6 and P8;
- to the left: m3, P4, P5, m6 and P8.



The greater harmoniousness between pairs of notes, or the consonance between more intervals than previously, means that in this system there is no room for compromise. On the contrary the usual semitone approximation disrupts the harmonic tensions of 5 intervals in the arc of an octave, 10 in the arc of two octaves, and so on. The acoustic difference is clearly perceptible to anyone like Capurso who tunes pianos by ear, not using an electronic tuner. He can "sort out" the first octave in 15 minutes (octave 4 on the keyboard) the first three octaves in an hour (octaves 3-4-5) and in a couple of hours, the whole instrument. It is also played for a while to adjust to the new tension-relationships and to enable any elastic hysteresis to emerge.

A piece played on a Chas-tempered (see www.chas.it) Steinway & Sons instrument creates a remarkable effect when listened to by the trained ear; even a simple scale plays in a more euphonic and totally natural way. Concert pianists hear the difference straight away, but others do as well.

The Harmonic Root

On the theoretical side, Capurso has found support from a group of mathematicians led by Prof. Filippo Spagnolo, at the University of Palermo. He also presented his findings to Benoit Mandelbrot himself, the father of fractals. The future may well bring the development of metamusical applications, built on the Chas system. In fact, harmony comes into being when in the whole we find the meaning of each single part and vice versa, just as is the case with Chas and with fractals. For reasons that require more time than we have here to explain, Chas points the way to a kind of 4D golden section in music, since sound is propagated in 3 spatial dimensions and also in time.

The **Harmonic Root,** or the "4:3 symmetric resonance harmonic constant" is produced from the following formula:

$$\boxed{(3 - \mathrm{D})^{\frac{1}{19}} = (4 + s\,\mathrm{D})^{\frac{1}{24}}}$$

Capurso explains the system's structure: Unlike previous systems, Chas does not give more importance to the 2:1 ratio over other ratios; instead, the system is based on an equal "difference-value", an identical distance of two quantities (3 and 4) from their pure value. This scale which is proportional in time finds its mathematical ratio in an "equi-difference". Each element of the Chas set gives up a small part of its "pure" harmonic value to be part of a "Large Harmonic Set", formed from the numbers 3 and 4 and the scale factors 19 and 24.

When the value of parameter s is established and the equation is solved as regards Δ, we obtain for both a constant factor k.

For example if s = 0, we find the semitone of the equal system:

$$(3 - \mathrm{D})^{\frac{1}{19}} = 2^{\frac{1}{12}} = k$$

$$\mathrm{D} \approx 0.00338585$$

$$k = \sqrt[12]{2} \approx 1.05946309$$

If $s = 1$, we obtain the Chas system semitone:

$$(3 - \mathrm{D})^{\frac{1}{19}} = (4 + \mathrm{D})^{\frac{1}{24}} = k$$

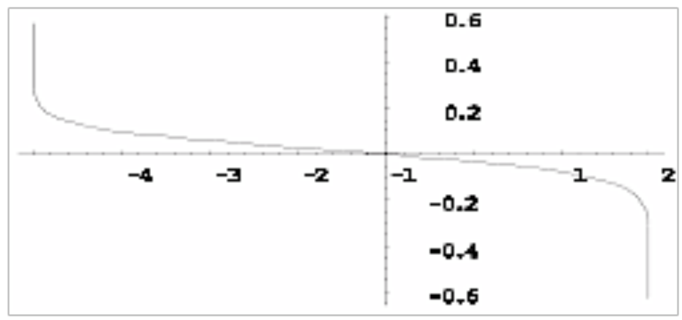
$$\mathrm{D} \approx 0.00212539$$

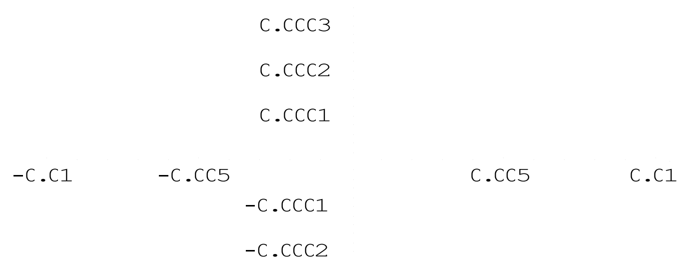
$$k \approx 1.05948654$$

Below is a graph of the function

$$f(\mathrm{D}) = (3 - \mathrm{D})^{\frac{1}{19}} - (4 + \mathrm{D})^{\frac{1}{24}}$$

and a zoom view in the neighbourhood of Δ=0.002.



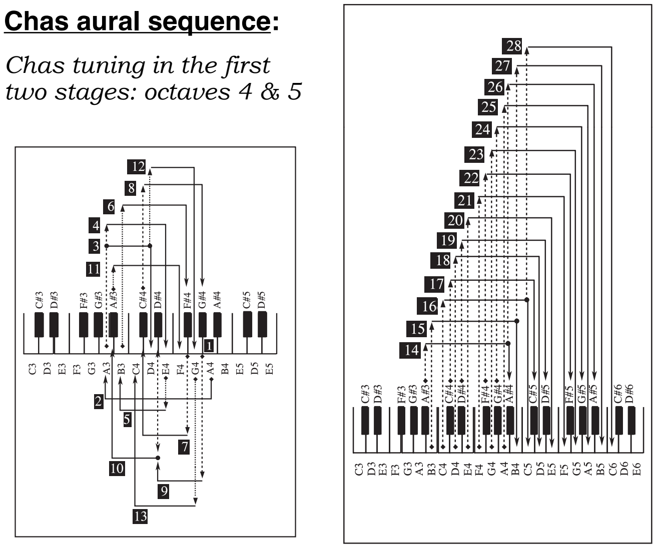


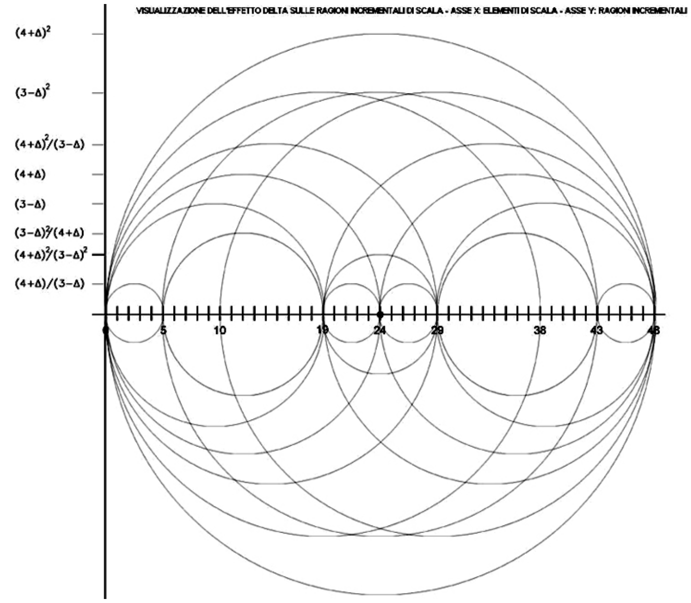
CHAS VS EQUAL TEMPERAMENT

Both systems, Chas and Equal Temperament use irrational proportions between the notes. But where Equal Temperament is based on approximations or compromise between the ideal sound and a sound that is easy to calculate, Chas is an ideal system where irrational numbers are used to eliminate approximations and thus create greater harmony.

Chas aural sequence:

Chas tuning in the first two stages: octaves 4 & 5





The differences between the frequencies in the two systems are minimal, but the underlying rationale of each, as we have seen, is profoundly different. If we look more closely at the numbers we discover that, if we express semitones in cents (3), the difference is around 0.04 cents per note: if tuning is begun with *La* at 440 Hz, *La* (first key on the pianoforte) will be only 0.03 Hz below the usual 27.5 Hz, while *Do*$_8$ (last key) will be 3.61 Hz above the usual 4186.01 Hz.

At the end of the process, the same ratio is established between the octaves differences as between the semitones.

We refer readers to the Chas website www.chas.it where the method is described in technical, theoretical and perceptual detail.

We hope that Capurso will find a modern-day Bach to embrace his brilliant system and make it known worldwide. Impossible? Definitely not, in the age of the worldwide web; it is the web which enabled us to learn about Chas. We are very happy to be able to present it to musicians and mathematicians.

■

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