EXPLICIT SOLUTION OF BURGERS' AND GENERALIZED BURGERS' EQUATION USING HOMOTOPY PERTURBATION METHOD

Miral Bhagat[1], P. H. Bhathawala[2] and Kaushal Patel[3]
[1]Asst. Prof., K.B.S. & Nataraj Science College, Vapi, Guj., India
[2]Professor & Former Head, Dept. of Mathematics, VNSGU, Surat, Guj., India
[3]Asst. Prof., Dept. of Mathematics, VNSGU, Surat, Guj., India

ABSTRACT

In recent years, many more of the numerical methods were used to solve a wide range of mathematical, physical and engineering problems, linear and nonlinear. In this article, Homotopy Perturbation Method (HPM) is employed to approximate the solution of the Burgers' equation which is a one-dimensional non-linear differential equation in fluid dynamics. The explicit solution of the Burgers' equation was obtained and compared with the exact solutions. We take the cases where the exact solution was not available for viscosity smaller than 0.01, we apply the HPM structure for obtaining the explicit solution. The results reveal that the HPM is very effective, convenient and quite accurate to partial differential equation.

KEYWORDS: *Homotopy Perturbation Method (HPM), Burgers' equation, Fluid dynamics, Kinematic viscosity*

I. INTRODUCTION

The HPM, first proposed by Ji-Huan He [8, 13], for solving differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, has a significant advantage that it provides an analytical approximate solution to a wide range of linear and nonlinear problems in applied sciences. The HPM is applied to Volterra's integro-differential equation [14], nonlinear oscillators [15], bifurcation of nonlinear problems [16], bifurcation of delay-differential equations [17], nonlinear wave equations [18], boundary value problems [19] and to other fields [20-28]. The HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions. Thus, He's HPM is a universal one which can solve various kinds of linear and nonlinear equations.

Now the recent development is to extend the application of the He's HPM to solve linear and nonlinear systems of partial differential equations such as the systems of coupled Burgers' equations in one- and two- dimension and the system of Laplace's equation [31].

In this paper, Homotopy Perturbation method has been explained. Then, analytical and numerical solutions of Burgers' and generalized Burgers' equation have been obtained using HPM. Comparison of analytical solution with numerical solution of Burgers' equation and generalized Burgers' equation are depicted at different times. Finally we have made concluding remarks based on the graphs.

The one dimensional non-linear partial differential equation is

$$\frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) = \vartheta\frac{\partial^2}{\partial x^2}u(x,t) \quad (1)$$

known as Burgers' equation. Burgers' model of turbulence is a very important fluid dynamics model and the study of this model and the theory of shock waves have been considered by many authors both for conceptual understanding of a class of physical flows and for testing various numerical methods. The distinctive feature of Eq. (1) is that it is the simplest mathematical formulation of the competition between non-linear advection and the viscous diffusion. It contains the simplest form of non-linear advection term uu_x and the dissipation term $\vartheta\ u_{xx}$ where $\vartheta = \frac{1}{Re}$ (ϑ: *kinematics viscosity* Re: *Reynolds number*) for simulating the physical phenomena of wave motion and thus determines the behaviour of the solution. The mathematical properties of Eq. (1) have been studied by Cole [1]. Particularly, the detailed relationship between Eq. (1) and both turbulence theory and the shock wave theory were described by Cole. He also gave an exact solution of Burgers' equation. Benton and Platzman [2] have demonstrated about 35 distinct exact solutions of Burgers' like equation and their classifications. It is well known that the exact solution of Burgers' equation can only be computed for restricted values of ϑ which represent the kinematics viscosity of the fluid motion. Because of this fact, various numerical methods were employed to obtain the solution of Burgers equation with small ϑ values.

Many numerical solutions for Eq. (1) have been adopted over the years. Finite element techniques have been employed frequently. For example, Varoglu and Finn [3] presented an isoparametric space B time finite-element approach for solving Burgers' equation, utilizing the hyperbolic differential equation associated with Burgers' equation. Another approach which has been used by Caldwell *et al.* [4] is the finite-element method such that by altering the size of the element at each stage using information from the previous steps.

Ã–zis *et al.* [5] applied a simple finite-element approach with linear elements to Burgers' equation reduced by Hopf-Cole transformation. Aksan and Özdes [6] have reduced Burgers' equation to the system of non-linear ordinary differential equations by discretization in time and solved each non-linear ordinary differential equation by Galerkin method in each time step. As they claimed, for moderately small kinematics viscosity, their approach can provide high accuracy while using a small number of grid points (i.e., N = 5) and this makes the approach very economical computational wise. In the case where the kinematics viscosity is small enough i.e., $\vartheta = 0.0001$, the exact solution is not available and a discrepancy exists, their results clarify the behavior of the solution for small times, i.e., $T = t_{max} \leq 0.15$. Also it is demonstrated that the parabolic structure of the equation decayed for $t_{max} = 0.5$. And finally, Aksan *et al.* [7] applied least squares method to find solution of this equation.

In this paper, the reduced Burgers' equation is solved by Homotopy Perturbation method. It is well known that the HPM converge very fast to the results. Moreover, contrary to the conventional methods which require the initial and boundary conditions, the HPM provide an analytical solution by using only the initial conditions. The boundary conditions can be used only to justify the obtained result. In the present study, it is aimed to establish the existence of the solution using the Homotopy Perturbation Method (HPM). Numerical examples are also presented for moderate values of ϑ since the exact solution is not available for lower values.

II. HOMOTOPY-PERTURBATION METHOD (HPM)

The fundamentals of Homotopy Perturbation Method, we consider the following non-linear differential equation:

$$A(u) - f(r) = 0, r \in \Omega \qquad (2)$$

with the boundary conditions of

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma \qquad (3)$$

where A, B, f(r) and Γ are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain **Ω**, respectively.

Generally speaking the operator A (u) can be divided into a linear part L(u) and a non-linear part N(u). Therefore eq. (2) can be rewritten as

$$L(u) + N(u) - f(r) = 0 \qquad (4)$$

By the Homotopy technique, we can construct $v(r,p): \Omega \times [0,1] \rightarrow R$, which satisfies:

$$H(v,p) = (1-p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0, \qquad (5)$$

where $p \in [0,1], r \in \Omega$,
OR

$$H(v,p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0 \qquad (6)$$

where p ϵ [0, 1] is an embedding parameter, while v_0 is an initial approximation of Eq. (2), which satisfies the boundary conditions. Obviously, from Eq. (5) and (6) we will have:

$$\text{For } p = 0, H(v,0) = L(v) - L(v_0) = 0 \qquad (7)$$

$$\text{For } p = 1, H(v,1) = A(v) - f(r) = 0 \qquad (8)$$

The changing process of p from zero to unity is just that of v(r, p) from $v_0(r)$ to v(r). In topology, this is called deformation while L (v) – L (v_0) and A (v) – f(r) are called Homotopy.

According to the HPM, we can first use the embedding parameter p as a small parameter and assume that the solution of Eq. (5) and (6) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 \dots\dots\dots \qquad (9)$$

Setting p = 1 yield in the approximate solution of Eq. (2) to:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 \dots\dots\dots \qquad (10)$$

The combination of the Perturbation method and the Homotopy method is called the Homotopy Perturbation Method, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages.

The series (10) is convergent for most cases. However, the convergence rate depends on the non-linear operator A(v).

Moreover following are the suggestions:

- The second derivative of N (v) with respect to v must be small because the parameter may be relatively large. i.e. p→1
- The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

The HPM has got many merits and it does not require small parameters in the equations, so that the limitations of the traditional perturbations can be eliminated. Also the calculations in the HPM are simple and straightforward. The reliability of the method and the reduction in the size of the computational domain gives this method a wider applicability.

III. ANALYTICAL SOLUTION OF BURGERS' EQUATION

Consider Burgers' equation (1) with the initial condition

$$u(x,0) = \sin \pi x \ , \quad 0 < x < 1 \qquad (11)$$

and homogeneous boundary conditions

$$u(0,t) = u(1,t) = 0 \ , \quad t > 0 \qquad (12)$$

By the Hopf–Cole transformation

$$u = -2\vartheta\left(\frac{\theta_x}{\theta}\right) \ , \qquad (13)$$

the Burgers' equation transforms to the linear heat equation

$$\frac{\partial\theta}{\partial t} = \vartheta \frac{\partial^2\theta}{\partial x^2} \qquad (14)$$

However, by the Hopf–Cole transformation, conditions (11) and (12) transform following conditions (15) and (16) respectively,

$$\theta_0(x) = \theta(x,0) = \exp\{-(2\vartheta\pi)^{-1}\{1-\cos(\pi x)\}\}, \ 0 < x < 1 \qquad (15)$$

$$\theta_x(0,t) = \theta_x(1,t) = 0, \quad t > 0 \qquad (16)$$

where if $\theta = \theta(x,t)$ is any solution of heat equation (14) then Eq. (13) is a solution of Burgers' equation (1) with the conditions (11) and (12).

Hence, using the method of separation of variables, the Fourier series solution to the above linearized problem, defined by Eqs. (14) – (16) can be obtained easily as

$$\theta(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\vartheta t}\cos(n\pi x) \qquad (17)$$

where a_0 and a_n (n = 1,2, ...) are Fourier coefficients and they are evaluated in the usual manner as

$$a_0 = \int_0^1 \exp\{-(2\vartheta\pi)^{-1}\{1-\cos(\pi x)\}\} \, dx \qquad (18)$$

$$a_n = 2\int_0^1 \exp\{-(2\vartheta\pi)^{-1}\{1-\cos(\pi x)\}\}\cos(n\pi x)\ dx \qquad (19)$$

(n = 1, 2, 3, ...)

Thus, using Eq. (13) the exact solution to the problem (1) is

$$u(x,t) = \frac{2\pi\vartheta\ \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\vartheta t}\, n\sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\vartheta t}\cos(n\pi x)} \qquad (20)$$

IV. HPM ALGORITHM FOR BURGERS' EQUATION

Let us consider Burgers' Eq. (1) with the initial and boundary conditions (11) & (12)
Separating the linear and non-linear parts of Eq. (1), we apply HPM to Eq. (5).
A Homotopy can be constructed as follows:

$$(1-p)\left[\frac{\partial}{\partial t}v(x,t) - \frac{\partial}{\partial x}v_0(x,t)\right] + p\left[\frac{\partial}{\partial t}v(x,t) + v(x,t)\frac{\partial}{\partial x}v(x,t) - \vartheta\frac{\partial^2}{\partial x^2}v(x,t)\right] = 0,$$

$$p \in [0,1] \qquad (21)$$

Substituting the value of v from Eq. (9) into (21) and rearranging based on powers of p-terms yields:

$$p^0: \frac{\partial}{\partial t}v_0(x,t) = 0 \qquad (22)$$

$$p^1: \left\{\frac{\partial}{\partial t}v_1(x,t)\right\} - \vartheta\left\{\frac{\partial^2}{\partial x^2}v_0(x,t)\right\} + v_0(x,t)\left\{\frac{\partial}{\partial x}v_0(x,t)\right\} = 0 \qquad (23)$$

$$p^2: \left\{\frac{\partial}{\partial t}v_2(x,t)\right\} - \vartheta\left\{\frac{\partial^2}{\partial x^2}v_1(x,t)\right\} + v_1(x,t)\left\{\frac{\partial}{\partial x}v_0(x,t)\right\} + v_0(x,t)\left\{\frac{\partial}{\partial x}v_1(x,t)\right\} = 0 \qquad (24)$$

$$p^3: \left\{\frac{\partial}{\partial t}v_3(x,t)\right\} - \vartheta\left\{\frac{\partial^2}{\partial x^2}v_2(x,t)\right\} + v_2(x,t)\left\{\frac{\partial}{\partial x}v_0(x,t)\right\} + v_0(x,t)\left\{\frac{\partial}{\partial x}v_2(x,t)\right\}$$
$$+ v_1(x,t)\left\{\frac{\partial}{\partial x}v_1(x,t)\right\} = 0 \qquad (25)$$

with the following conditions

$$v_0(x,0) = \sin(\pi x), v_0(0,t) = 0, v_0(1,t) = 0,$$
$$v_i(x,0) = 0, v_i(0,t) = 0, v_i(1,t) = 0, i = 1,2 \ldots. \qquad (26)$$

The solutions of Eq. (22)-(25) by using the conditions (26), may be re-written as follows:

$$v_0(x,t) = \sin(\pi x), \qquad (27)$$

$$v_1(x,t) = -t\,\vartheta\sin(\pi x)\,\pi^2 - \frac{1}{2}t\pi\sin(2\pi x), \qquad (28)$$

$$v_2(x,t) = \frac{1}{8}\pi^2 t^2\{4\vartheta^2\sin(\pi x)\,\pi^2 + 12\vartheta\pi\sin(2\pi x) + 3\sin(3\pi x)$$
$$- \sin(\pi x)\}, \qquad (29)$$

$$v_3(x,t) = -\frac{1}{24}\pi^3 t^3\{56\vartheta^2\sin(2\pi x)\,\pi^2 + 51\vartheta\pi\sin(3\pi x) - 9\vartheta\pi\sin(\pi x) + 6\sin(4\pi x) - 4\sin(2\pi x)$$
$$+ 4\vartheta^3\pi^3\sin(\pi x)\}, \qquad (30)$$

Similarly, the other components can be obtained.

Substituting Eq. (27)-(30) into (10), then re-written as follows:

$$u(x,t) = \sin(\pi x) - t\,\vartheta\sin(\pi x)\,\pi^2 - \frac{1}{2}t\pi\sin(2\pi x)$$
$$+ \frac{1}{8}\pi^2 t^2\{4\varepsilon^2\sin(\pi x)\,\pi^2 + 12\vartheta\pi\sin(2\pi x) + 3\sin(3\pi x) - \sin(\pi x)\}$$
$$- \frac{1}{24}\pi^3 t^3\{56\vartheta^2\sin(2\pi x)\,\pi^2 + 51\vartheta\pi\sin(3\pi x) - 9\vartheta\pi\sin(\pi x) + 6\sin(4\pi x)$$
$$- 4\sin(2\pi x) + 4\vartheta^3\pi^3\sin(\pi x)\} \qquad (31)$$

V. HPM ALGORITHM FOR GENERALIZED BURGERS' EQUATION

Generalized Burgers' Equation is

$$u_t = \vartheta u_{xx} - u u_x + u \tag{32}$$

Linear part = $u_t - u$ and Non Linear part = $u u_x$

$$H(v,p) = (1-p)\left[\frac{\partial}{\partial t} v(x,t) - v(x,t) - \left\{\frac{\partial}{\partial t} v_0(x,t) - v_0(x,t)\right\}\right] + p\left[\frac{\partial}{\partial t} v(x,t) + v(x,t)\frac{\partial}{\partial x} v(x,t) - \vartheta \frac{\partial^2}{\partial x^2} v(x,t) - v(x,t)\right] = 0$$

Substituting Eqn. (9) in H(v, p) and rearranging based on powers of p-terms yields:

$$p^0 : \frac{\partial v_0(x,t)}{\partial t} = 0 \tag{33}$$

$$p^1 : \frac{\partial v_1}{\partial t} - v_1 + \frac{\partial v_0}{\partial t} + v_0\frac{\partial v_0}{\partial x} - v_0 - \vartheta\frac{\partial^2 v_0}{\partial x^2} \tag{34}$$

$$p^2 : \frac{\partial v_2}{\partial t} - v_2 + v_1\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_1}{\partial x} - \vartheta\frac{\partial^2 v_1}{\partial x^2} \tag{35}$$

$$p^3 : \frac{\partial v_3}{\partial t} - v_3 + v_1\frac{\partial v_1}{\partial x} + v_0\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_0}{\partial x} - \vartheta\frac{\partial^2 v_2}{\partial x^2} \tag{36}$$

The solutions of Eq. (33)-(36), may be re-written as follows:

$$\frac{\partial v_0}{\partial t} = 0 \tag{37}$$

$$\frac{\partial v_1}{\partial t} = v_1 - \frac{\partial v_0}{\partial t} - v_0\frac{\partial v_0}{\partial x} + v_0 + \vartheta\frac{\partial^2 v_0}{\partial x^2} \tag{38}$$

$$\frac{\partial v_2}{\partial t} = v_2 - v_1\frac{\partial v_0}{\partial x} - v_0\frac{\partial v_1}{\partial x} + \vartheta\frac{\partial^2 v_1}{\partial x^2} \tag{39}$$

$$\frac{\partial v_3}{\partial t} = v_3 - v_1\frac{\partial v_1}{\partial x} - v_0\frac{\partial v_2}{\partial x} - v_2\frac{\partial v_0}{\partial x} + \vartheta\frac{\partial^2 v_2}{\partial x^2} \tag{40}$$

with following conditions

$v_0(x,0) = \sin(\pi x), v_0(0,t) = 0, v_0(1,t) = 0, v_i(x,0) = 0, v_i(0,t) = 0, v_i(1,t) = 0, i = 1,2 ....$

So, we have

$$v_0(x,t) = \sin \pi x \tag{41}$$

For v_1 (x, t), using eqn. (38) and (28)

$$\frac{\partial v_1}{\partial t} - v_1 = v_0 - \frac{\partial v_0}{\partial t} + v_0\frac{\partial v_0}{\partial x} - \vartheta\frac{\partial^2 v_0}{\partial x^2}$$

$$\frac{\partial v_1}{\partial t} = v_1 + \sin \pi x - \frac{\partial \sin \pi x}{\partial t} - \sin \pi x\frac{\partial \sin \pi x}{\partial x} + \vartheta\frac{\partial^2 \sin \pi x}{\partial x^2}$$

$$\frac{\partial v_1}{\partial t} = -\pi^2 t\,\vartheta \sin(\pi x) - \frac{1}{2} t\pi \sin(2\pi x) + \sin \pi x - \frac{\pi \sin 2\pi x}{2} - \vartheta\,\pi^2 \sin \pi x$$

Integrating with respect to "t" we get

$$v_1 = t \sin \pi x - \frac{t^2 \pi \sin 2\pi x}{2} - \frac{\pi t \sin 2\pi x}{2} - \vartheta\,\pi^2 t \sin \pi x - \frac{\vartheta \pi^2 t^2 \sin \pi x}{2} \tag{42}$$

Similarly,

$$v_2 = \frac{2\pi^4 t^3 \vartheta^2 \sin \pi x}{3} + \frac{4\pi^3 t^3 \vartheta \sin 2\pi x}{3} + \frac{3\pi^3 t^2 \vartheta \sin 2\pi x}{2} - \frac{\pi^2 t^2 \vartheta \sin \pi x}{2} + \frac{3\pi^2 t^3 \sin 3\pi x}{8} - \frac{\pi^2 t^3 \sin \pi x}{8} + \frac{3\pi^2 t^2 \sin 3\pi x}{8} - \frac{\pi t^2 \sin 2\pi x}{2} - \frac{\pi^2 t^2 \sin \pi x}{8} \quad (43)$$

and

$$v_3 = -\frac{\pi t^2 \sin 2\pi x}{4} + \frac{\vartheta\pi^4 t^3 \sin\pi x}{3} - \frac{17\vartheta\pi^4 t^3 \sin 3\pi x}{8} + \frac{\vartheta\pi^3 t^3 \sin 2\pi x}{3} - \frac{3\pi^3 t^3 \sin\pi x \cos 3\pi x}{8} + \frac{2\pi^2 t^3 \sin\pi x \cos 2\pi x}{3} + \frac{\pi^3 t^3 \sin 2\pi x}{24} + \frac{\pi^2 t^3 \sin 2\pi x \cos\pi x}{3} - \frac{\pi^3 t^3 \sin 2\pi x \cos 2\pi x}{6} + \frac{\vartheta\pi^2 t^3 \sin\pi x}{12} - \frac{\vartheta\pi^5 t^3 \sin 2\pi x}{6} - \frac{\pi^3 t^3 \sin 3\pi x \cos\pi x}{8} + \frac{\vartheta^2\pi^4 t^3 \sin\pi x}{6} - \frac{2\vartheta\pi^3 t^3 \sin 2\pi x}{3} + \frac{\vartheta\pi^4 t^3 \sin 2\pi x}{3} - \frac{7\vartheta\pi^5 t^4 \sin 2\pi x}{12} - \frac{11\vartheta\pi^4 t^4 \sin 3\pi x}{8} + \frac{\vartheta\pi^4 t^4 \sin\pi x}{8} - \frac{\pi^3 t^4 \sin 4\pi x}{4} + \frac{7\pi^3 t^4 \sin 2\pi x}{96} - \frac{5\vartheta^3\pi^6 t^4 \sin\pi x}{24} - \frac{13\vartheta^2\pi^5 t^4 \sin 2\pi x}{8} - \frac{9\pi^3 t^4 \sin\pi x \cos 3\pi x}{32} - \frac{3\pi^3 t^4 \sin 3\pi x \cos\pi x}{32} - \frac{\pi^2 t^4 \sin\pi x}{16} + \frac{3\pi^2 t^4 \sin 3\pi x}{16} + \frac{3\vartheta\pi^3 t^4 \sin 2\pi x}{16} - \frac{25\pi^4 t^4 \sin\pi x \cos 2\pi x}{24} - \frac{31\vartheta\pi^4 t^4 \sin 2\pi x \cos\pi x}{48} - \frac{\pi^3 t^5 \sin 4\pi x}{10} + \frac{\vartheta\pi^4 t^5 \sin\pi x}{40} - \frac{3\vartheta\pi^4 t^5 \sin 3\pi x}{40} - \frac{\vartheta^2\pi^5 t^5 \sin 2\pi x}{40} \quad (44)$$

Similarly, the other components can be obtained.
Substituting eqns. (41) to (44) in equation (10) we get

$$u(x,t) = \sin\pi x + \pi t \sin\pi x - \pi t \sin\pi x - \frac{\pi t \sin\pi x}{2} - \vartheta\pi^2 t \sin\pi x - \frac{5\pi t^2 \sin 2\pi x}{4} - \vartheta\pi^2 t^2 \sin\pi x + \frac{3\vartheta\pi^3 t^2 \sin 2\pi x}{2} + \frac{3\pi^2 t^2 \sin 3\pi x}{8} - \frac{\pi^2 t^2 \sin\pi x}{8} + \frac{5\vartheta^2\pi^4 t^3 \sin\pi x}{6} + \frac{7\pi^2 t^3 \sin 3\pi x}{8} - \frac{7\pi^2 t^3 \sin\pi x}{24} + \vartheta\pi^3 t^3 \sin 2\pi x - \frac{3\pi^3 t^3 \sin\pi x \cos 3\pi x}{8} + \frac{\pi^3 t^3 \sin 2\pi x}{24} - \frac{\pi^3 t^3 \sin 2\pi x \cos 2\pi x}{6} + \frac{\vartheta\pi^4 t^3 \sin\pi x}{3} - \frac{17\vartheta\pi^4 t^3 \sin 3\pi x}{8} - \frac{\vartheta\pi^5 t^3 \sin 2\pi x}{6} - \frac{\pi^3 t^3 \sin 3\pi x \cos\pi x}{8} + \frac{\vartheta\pi^4 t^3 \sin 2\pi x}{3} - \frac{7\vartheta\pi^5 t^4 \sin 2\pi x}{12} - \frac{11\vartheta\pi^4 t^4 \sin 3\pi x}{8} + \frac{\vartheta\pi^4 t^4 \sin\pi x}{8} - \frac{\pi^3 t^4 \sin 4\pi x}{4} + \frac{7\pi^3 t^4 \sin 2\pi x}{96} - \frac{5\vartheta^3\pi^6 t^4 \sin\pi x}{24} - \frac{13\vartheta^2\pi^5 t^4 \sin 2\pi x}{8} - \frac{9\pi^3 t^4 \sin\pi x \cos 3\pi x}{32} - \frac{3\pi^3 t^4 \sin 3\pi x \cos\pi x}{32} - \frac{\pi^2 t^4 \sin\pi x}{16} + \frac{3\pi^2 t^4 \sin 3\pi x}{16} + \frac{3\vartheta\pi^3 t^4 \sin 2\pi x}{16} - \frac{25\pi^4 t^4 \sin\pi x \cos 2\pi x}{24} - \frac{31\vartheta\pi^4 t^4 \sin 2\pi x \cos\pi x}{48} - \frac{\pi^3 t^5 \sin 4\pi x}{10} + \frac{\vartheta\pi^4 t^5 \sin\pi x}{40} - \frac{3\vartheta\pi^4 t^5 \sin 3\pi x}{40} - \frac{\vartheta^2\pi^5 t^5 \sin 2\pi x}{40} \quad (45)$$

VI. COMPARISON OF HPM SOLUTIONS WITH EXACT SOLUTIONS

In order to demonstrate the adoptability and accuracy of the present approaches, we have applied it to the problem given by Eq. (1) whose exact solution exists and is given by Cole [1] in terms of infinite series. To emphasize the accuracy of the method for moderate size viscosity values, we give the comparison with analytical solutions obtained from the infinite series of Cole for $\vartheta = 1$ in Table 1, 2 & 3, which shows that the solutions are in good agreement with analytical solutions. In the following tables, HPM solutions are obtained for $\vartheta = 1$ at different times. In the case ϑ is smaller than 0.01, the exact solution is not available and a discrepancy exists. Also, it is not practical to evaluate the analytical solution at these values due to slow convergence of the infinite series and thus the exact solution in this regime is unknown.

6.1 Figures and Tables

Table 1: Analytical solutions of Burgers' equation obtained at t = 0.0001, t = 0.001, t = 0.01 and ϑ = 1

x	t = 0.0001	t = 0.001	t = 0.01
0	0	0	0
0.1	0.3072	0.3037	0.2724
0.2	0.5861	0.5797	0.5210
0.3	0.8098	0.8012	0.7228
0.4	0.9549	0.9455	0.8575
0.5	1.0054	0.9962	0.9096
0.6	0.9551	0.9473	0.8714
0.7	0.8098	0.8039	0.7451
0.8	0.5860	0.5822	0.5430
0.9	0.3070	0.3052	0.2859
1	1.22E-16	1.21E-16	1.13E-16

Table 2: Numerical solutions of Burgers' equation obtained at t = 0.0001, t = 0.001, t = 0.01 and ϑ = 1

x	t = 0.0001	t = 0.001	t = 0.01
0	0	0	0
0.1	0.3086	0.3051	0.2732
0.2	0.5871	0.5806	0.5214
0.3	0.8081	0.7996	0.7218
0.4	0.9500	0.9408	0.8546
0.5	0.9990	0.9902	0.9058
0.6	0.9502	0.9426	0.8684
0.7	0.8084	0.8025	0.7441
0.8	0.5874	0.5835	0.5438
0.9	0.3088	0.3069	0.2870
1	1.22E-16	1.22E-16	1.14E-16

Table 3: Numerical solutions of Gen. Burgers' equation obtained at t = 0.0001, t = 0.001, t = 0.01 and ϑ = 1

x	t = 0.0001	t = 0.001	t = 0.01
0	0	0	0
0.1	0.3087	0.3054	0.2759
0.2	0.5871	0.5811	0.5266
0.3	0.8082	0.8004	0.7289
0.4	0.9501	0.9418	0.8631
0.5	0.9991	0.9912	0.9148
0.6	0.9503	0.9436	0.8771
0.7	0.8084	0.8033	0.7516
0.8	0.5874	0.5840	0.5493
0.9	0.3088	0.3072	0.2899
1	1.22E-16	1.22E-16	1.15E-16

Table 4: Comparison of solutions obtained at t = 0.01 and ϑ = 1

x	Analytical	Burgers'	Gen. Burgers'
0	0	0	0
0.1	0.2724	0.2732	0.2759
0.2	0.5210	0.5214	0.5266
0.3	0.7228	0.7218	0.7289
0.4	0.8575	0.8546	0.8631
0.5	0.9096	0.9058	0.9148
0.6	0.8714	0.8684	0.8771

0.7	0.7451	0.7441	0.7516
0.8	0.5430	0.5438	0.5493
0.9	0.2859	0.2870	0.2899
1	1.13E-16	1.14E-16	1.15E-16

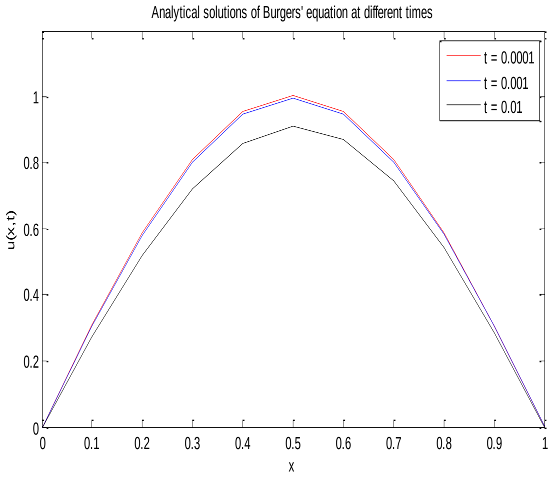

Figure 1: Comparison of analytical solutions of Burgers' equation at different times when $\vartheta = 1$.

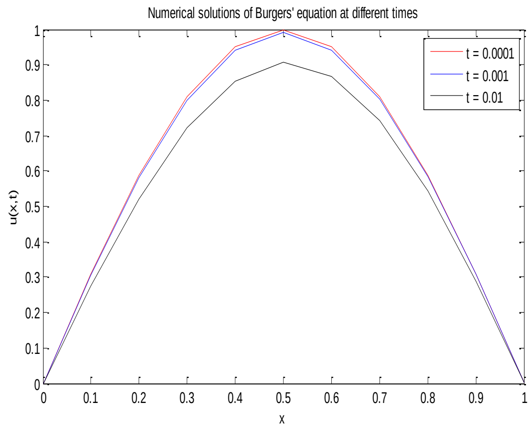

Figure 2: Comparison of numerical solutions of Burgers' equation by HPM at different times when $\vartheta = 1$.

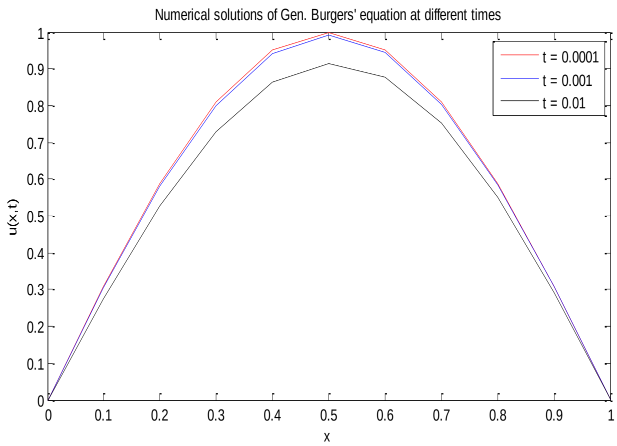

Figure 3: Comparison of numerical solutions of Gen. Burgers' equation by HPM at different times when $\vartheta = 1$.

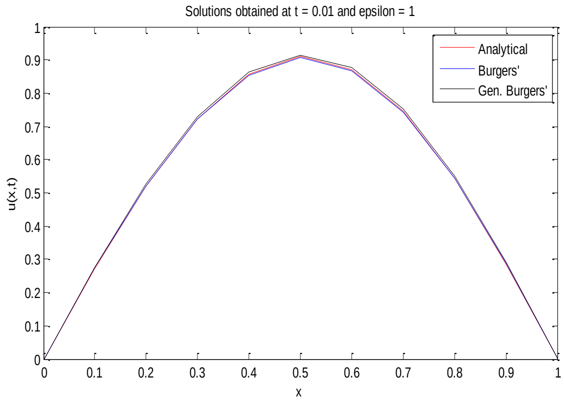

Figure 4: Comparison of analytical solution of Burgers' equation with numerical solution of Burgers' and Generalized Burgers' equation by HPM at t = 0.01and ϑ = 1.

VII. CONCLUSIONS

The Homotopy Perturbation method has been successfully applied to the Burgers' equations and generalized Burgers' equation. It provides more realistic series solutions that converge very rapidly in real physical problems. Therefore, this method is a powerful mathematical tool for solving Burgers' equation and generalized Burgers' equation. Also it is very effective, convenient and quite accurate to systems of partial differential equations. The obtained results show that the method is also a promising method to solve other linear and nonlinear partial differential equations.

VIII. FUTURE WORK

This work can be extended to increase the spatial dimension of the equation that is considered in this paper. Our attempt is to develop the explicit iterative algorithm of the equations for finding solutions for the subsequent time.

REFERENCES

[1] Cole J. D., (1951), *On a quasilinear parabolic equation occurring in aerodynamics*, Quart. Appl. Math., 9, pp 225-236.
[2] Benton E. R., Platzman G. W., (1972) *A table of solution of the one dimensional Burgers' equation*, Quart. Appl. Math., 30, pp 195-212.
[3] Varoglu E. & Finn W. D. L., (1980), *Space B time finite elements incorporating characteristics for the Burgers' equation*, Int. J. Num. Math. Eng., 16, pp 171-184.
[4] Caldwell J., Wanless P. & Cook a. E., (1981), *A finite element approach to Burgers' equation*, Applied Math Model, 5, pp 189-193.
[5] Ã–zis, T., E.N. Aksan and A. Ã–zdes, (2003) *A finite element approach for solution of Burgers' equation*, Applied Math. Comput., 139, pp 417-428.
[6] Aksan, E.N. and A. Ã–zdes, (2002) *A numerical solution of Burgers' equation*, Applied Math. Comput., 156, pp 395-402.
[7] Aksan, E.N., A. Ã–zdes and T. Ã–zis, (1972) *A numerical solution of Burgers' equation based on least squares approximation*, Applied Math. Comput., 176, pp 270-279.
[8] He J. H., (1999), *Homotopy Perturbation technique*, Comput. Methods Applied Mech. Eng., 178, pp 257-262.
[9] He J. H., (2006), *New interpretation of Homotopy Perturbation method*, Int. J. Modern Phys. B., 20, pp 2561-2568.
[10] Noorzad R., Tahmasebi Poor A. & Omidvar M., (2008), *Variational iteration method & Homotopy Perturbation method for solving Burgers' equation in fluid dynamics*, J. of Appl. Sciences, 8, pp 369-373.
[11] E. Hopf, (1950), *The partial differential equation* $u_t + uu_x = \mu u_{xx}$, Commun. Pure Appl. Math. 3(3),pp 201-230.
[12] Burgers J. M. (1939), *Mathematical examples illustrating relations occurring in the theory ofturbulent fluid motion*, Trans. Roy. Neth. Acad. Sci. Amsterdam, 17, pp 1-53.

[13] He J. H., (2000), *A coupling method of a homotopy technique and a perturbation technique for non-linear problems*, Int. Jour. of Non-linear mechanics, 35, pp 37-43.
[14] El-Shahed M., (2005), *Application of He's homotopy–perturbation method to Volterra's integro-differential equation*, Int. J. Non-linear Sci. Numer. Simul., 6 (2), pp 163.
[15] He J. H., (2004), *The Homotopy Perturbation method for non linear oscillators with discontinuities*, J. Appl. Math. Comput., 151, pp 287-292.
[16] He J. H., (2005), *Homotopy Perturbation method for bifurcation of non linear problems*, Int. J. Non-linear Sci. Numer. Simul., 6 (2), pp 207-208.
[17] He J. H., (2005), *Periodic solutions and bifurcations of delay-differential equations*, Phys. Lett. A 374 (4-6), pp 228.
[18] He J. H., (2005), *The use of adomian decomposition method for solving the regularized long-wave equation*, Chaos Solitons Fractals, 26 (3), pp 695.
[19] He J. H., (2006), *Homotopy Perturbation method for solving boundary value problems*, Phys. Lett. A 350 (1-2), pp 87.
[20] He J. H., (2003), *Homotopy Perturbation method: a new non linear analytical technique*, J. Appl. Math. Comput., 135, pp 73-79.
[21] He J. H., (2004), *Comparison of Homotopy Perturbation method and Homotopy Analysis method*, J. Appl. Math. Comput., 156, pp 527-539.
[22] He J. H., (2004), *Asymtotology by Homotopy Perturbation method*, J. Appl. Math. Comput., 156, pp 591-596.
[23] He J. H., (2005), *Application of Homotopy Perturbation method to non linear wave equations*, Chaos Solitons Fractals, 26 (3), pp 827.
[24] He J. H., (2006),*New Interpretation of Homotopy Perturbation method*, Int. J. Mod. Phys. B 20 (10), pp 1141.
[25] Siddiqui A. M., Mahmood R., Ghori Q. K., (2006), *Thin film flow of a third grade fluid on a moving belt by He's Homotopy Perturbation method*, Int. J. Non-linear Sci. Numer. Simul. 7(1), pp 7.
[26] Siddiqui A. M., Mahmood R., Ghori Q. K., (2006), *Couette and poiseuille flows for non-newtonian fluids*, Int. J. Nonlinear Sci. Numer. Simul. 7 (1) (2006) 15.
[27] Abbasbandy S., (2006), *Homotopy Perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method*, Appl. Math. Comput. 172 485-490.
[28] Abbasbandy S., (2006), *Numerical solutions of the integral equations: Homotopy perturbation method and Adomian's decomposition method*, Appl. Math. Comput. 173 493-500.
[29] Momani S., Odibat Z., (2007), *Homotopy Perturbation method for non linear partial differential equations of fractional order*, Phys. Lett. A, 365 (5-6), pp 345-350.
[30] Anderson D. A., Tannehill J. C., Pletcher R. H., (1984), *Computational Fluid Mechanics and Heat Transfer*, Hemisphere Publishing Corporation, Washington New York London, McGraw-Hill Book Company.
[31] Hemeda A. A., (2012), *Homotopy Perturbation Method for Solving Systems of Nonlinear Coupled Equations, Applied Mathematical Sciences*, Vol. 6, no. 96, pp 4787 – 4800.

AUTHORS

Miral A. Bhagat is working as an Assistant Professor at K. B. S. Commerce & Nataraj Professional Sciences College, Vapi, Gujarat, since 3 years. She is pursuing Ph. D. under the guidance of Dr. P. H. Bhathawala and co guidance of Dr. Kaushal B. Patel. Her subjects of interest are Discrete Mathematics & Fluid Dynamics.

Pravin H. Bhathawala is a Retired Professor and Head of the Department of Mathematics, Veer Narmad South Gujarat University, Surat, Gujarat, India. He has served the department for 32 years. He has qualified about 15 Ph. D. and 18 M. Phil. students under his guidance. His area of interest is Biomedical Mathematics and Fluid dynamics of porous media. Presently he is working with Babaria Institute of Science and Technology, Vadodara, Gujarat.

Kaushal B. Patel is working as an Assistant Professor at Department of Mathematics, Veer Narmad South Gujarat University, Surat, Gujarat, India, since 6 years. His area of interest is Computational Fluid Dynamics and Mathematical Modelling.