LINEAR ALGEBRA

Jobayer Al-Mahmud

Matrix: Rectangular arrangement of numbers is called matrix. Numbers are called elements or entries of that matrix. A matrix with m rows and n columns is called an m by n matrix & written as $\times n$ i.e. $row \times column$

A matrix usually presented in the following form-

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & \dots & \dots & a_{mn} \end{bmatrix}$$

The rows of A are horizontal lists of numbers. As-

$$(a_{11},\ a_{12},\ a_{13}, \dots \dots a_{1n})\ (a_{21},\ a_{22},\ a_{23}, \dots \dots a_{2n}) \dots \dots\ (a_{m1},\ a_{m2},\ a_{m3}, \dots \dots a_{mn})$$

And the columns of A are vertical lists of numbers. As-

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \\ \dots \\ a_{m3} \end{bmatrix} \dots \dots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}$$

Element a_{ij} is called ij-entry or element appears in row i and column j.
We denote such a matrix by simply writing A=[a_{ij}]

Two matrix A and B are equal, which is written as $A = B$, if they have same size and if corresponding elements are equal.

Addition and Subtraction of Matrix: Let, A and B be two matrices. Then addition of A and B, that is $A + B$ is permissible if for every elements of A there is a corresponding element in B. If there is no corresponding entry for every element of A, then $A + B$ is not permissible. Addition of A and B is same as addition of B and A. i.e. $A + B = B + A$

Subtraction of A and B, that is $A - B$ is permissible if for every elements of A there is a corresponding element in B. If there is no corresponding entry for every element of A, then $A - B$ is not permissible. Subtraction of A and B is not same as subtraction of B and A. That is $A - B \neq B - A$

Example:

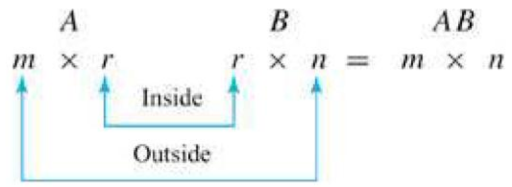
Let, A= $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, B= $\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$, C = $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}$

Then $A + B$ and $A - B$ is permissible. As for every entries of A there is a corresponding entry in B. But $A + C$, $B + C$ or $A - C$, $B - C$ are not permissible.

$$A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 14 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$$

Multiplication of Matrix: Let, $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices such that $O(A) = m \times r$ and $O(B) = r \times n$. Then the multiplication of A and B is permissible or defined and written as $A \times B = (C_{ij})_{m\times n}$



> *Multiplication of matrix doesn't follow the commutative laws. i.e. A x B ≠ B x A*
>
> *Multiplication = First matrix horizontal entries x Second matrix vertical entries*

Example:

Let, $A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}_{2\times2}$ & $B = \begin{pmatrix} 5 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix}_{2\times3}$

$$A \times B = \begin{pmatrix} 10+8 & 0+12 & 2+16 \\ 15+2 & 0+3 & 3+4 \end{pmatrix} = \begin{pmatrix} 18 & 12 & 18 \\ 17 & 3 & 7 \end{pmatrix}$$

Square Matrix: When the number of rows and columns of a matrix is equal called square matrix.

Example: $\begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$ is a square matrix of order 2,

$\begin{pmatrix} 1 & 2 & 4 \\ 5 & 4 & 5 \\ 3 & 0 & 4 \end{pmatrix}$ is a square matrix of order 3

Identity Matrix: A square matrix is called an Identity matrix if A=(a_{ij})

$$\text{Where } a_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

$I_1 = (1)$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Inverse of a Matrix: Let, A be a square matrix of order n, then a matrix B of same order of A is called inverse of A if $AB = BA = I_n$

If such B exists, then A is called invertible. We can write $B = A^{-1}$

Transpose Matrix: Transpose of a matrix A is written as A^T or A' is obtained by writing the columns of A in order as row.

Example: If $A = \begin{pmatrix} 5 & -1 \\ 2 & 6 \end{pmatrix}$, then transpose of A is $A^T = \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix}$

If $A = \begin{pmatrix} 1 & 3 & 5 \\ 4 & 2 & 7 \end{pmatrix}$ then transpose of A is $A^T = \begin{pmatrix} 1 & 4 \\ 3 & 2 \\ 5 & 7 \end{pmatrix}$

Let A and B be square matrices of the same size. If is invertible, then A and B must also be invertible.

Theorem: *If the sizes of the matrices are such that the stated operations can be performed, then-*

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(kA)^T = kA^T$
4. $(A\,B)^T = B^T A^T$

$$(AB)^T = B^T A^T$$

From matrix multiplication formula we know, $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

$$\text{Now, } \left[(AB)_{ji}\right]^T = (AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$
$$= \sum_{k=1}^{n} (A_{ki})^T . \left(B_{jk}\right)^T$$
$$= \sum_{k=1}^{n} \left(B_{jk}\right)^T . (A_{ki})^T$$

The product on the right is the $(j,i)th$ entry of $B^T A^T$,

while $\left[(AB)_{ji}\right]^T$ is the $(j,i)th$ entry of $(AB)^T$.

Therefore $(AB)^T = B^T A^T$ since their corresponding entries are equal.

Singular Matrix: A square matrix A is called singular if $|A|$=0, otherwise its called non-singular. Where $|A|$ is the determinate of A.

A matrix A is invertible if it is non-singular

$$\text{Let, } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Now if $|A| = ad - bc \neq 0$, A is invertible. But if $|A| = ad - bc = 0$, A is not invertible.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix},$$

Then $|A| = 3 + 2 = 5 \neq 0$

$$A^{-1} = \frac{1}{|5|}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

Check:

$$A\,A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= I$

Minor: If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ then the minor of ij is M_{ij}

$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ etc.

Co-factor: Let $A = [a_{ij}]$ be a square matrix of order *n*, then co-fector of a_{ij} is

$$\alpha_{ij} = (-1)^{i+j} M_{ij}$$

Example: If $A = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 4 & 5 \\ 3 & 0 & 4 \end{pmatrix}$ then A_{12} = -5, A_{32}= 15 etc.

Ad-joint Matrix: Let $A = [a_{ij}]$ be a square matrix of order n and cofactor of a_{ij} is A_{ij}. Now adjoint matrix of A, $adj(A) = [A_{ij}]^T$

Example 1*:* Find ad-joint matrix of A where $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$

Cofactors of A are $A_{11} = -18$, $A_{12} = 2$, $A_{13} = 4$

$A_{21} = -11$, $A_{22} = 14$, $A_{23} = 5$

$A_{31} = -10$, $A_{32} = -4$, $A_{33} = -8$

$$Adj\,(A) = \begin{pmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{pmatrix}$$

Ad-joint Matrix এর ক্ষেত্রে প্রথমে cofactor বের করতে হবে, তারপর তাকে transpose করতে হবে

Example 2*:* Find inverse matrix of A, where $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

Here, $|\,A\,| = 5 + 1 + 6 = 12 \neq 0$

As $|\,A\,|$ is not equals to zero, so A is invertible.

Cofactors of A are $A_{11} = 5$, $A_{12} = -1$, $A_{13} = 3$

$A_{21} = 4$, $A_{22} = 4$, $A_{23} = 0$

$A_{31} = -7$, $A_{32} = -1$, $A_{33} = 3$

Ad-joint matrix of A is $Adj(A) = \begin{pmatrix} 5 & 4 & -7 \\ -1 & 4 & -1 \\ 3 & 0 & 3 \end{pmatrix}$

Inverse of A is $A^{-1} = \frac{1}{12}\begin{pmatrix} 5 & 4 & -7 \\ -1 & 4 & -1 \\ 3 & 0 & 3 \end{pmatrix}$

Check: $A\,A^{-1} = \frac{1}{12}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 & -7 \\ -1 & 4 & -1 \\ 3 & 0 & 3 \end{pmatrix}$

$= \frac{1}{12}\begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= I_3$

Theorem: *Consider any same size matrix A, B, C and any scalars k and k′ then-*

1. $A + (B + C) = (A + B) + C$
2. $A + (-A) = (-A) + A = 0$
3. $A + 0 = 0 + A = A$
4. $A + B = B + A$
5. $k(A + B) = kA + kB$
6. $(k + k')A = kA + k'A$
7. $(kk')A = k(k'A)$
8. $1.A = A$

Diagonal & Trace: Let A=$[a_n]$ be an n square matrix. The diagonal or main diagonal of A consists of the elements with the same subscripts.

That is $a_{11}, a_{22}, a_{33}, \dots \dots \dots\ a_{nn}$

The trace of A, written as $tr(A)$ is the sum of the diagonal elements. Normally-

$$tr\ (A) = a_{11} + \ a_{22} + \ a_{33} + \ \dots \dots \dots + a_{nn}$$

Although $AB \neq BA$ but there trace $tr(AB)$ & $tr(BA)$ are equal

Diagonal হচ্ছে square matrix এর কর্ণ এবং Trace হচ্ছে ঐ কর্ণের entry গুলোর যোগফল

Theorem: *Let A= $[a_n]$, B=$[b_n]$ are n square matrix and k be any scalar. Then-*

1. $tr\ (A + B) = tr(A) + tr(B)$
2. $tr\ (k.A) = k.tr(A)$
3. $tr\ (A^T) = tr\ (A)$
4. $tr\ (AB) = tr(BA)$

Powers of Matrices: Let A be a n *square* matrix over a field k. Then powers of A are defined as follows-

$$A^0 = I, \quad A^1 = A, \quad A^2 = A.A, \quad A^3 = A.A.A, \quad A^{n+1} = A^n.A$$

Moreover if A is invertible then-

$A^{-n} = (A^{-1})^n$

$= A^{-1}.A^{-1}.A^{-1} \dots\ A^{-1}$ (n factors, n>1)

Polynomial of Matrices: Let A be a *square* matrix & $P(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n$ then $P(A) = \ a_0I + a_1A + a_2A^2 + \cdots a_nA^n$ is called the polynomial of A of degree n.

Example: If $P(x) = 2x^3 + 3x + 1$ and $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ then $P(A) =?$

$$2\ A^2 = 2\ A.A = \ 2\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = 2\begin{pmatrix} -1 & 7 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} -2 & 14 \\ -8 & 14 \end{pmatrix}$$

$$3A = 3\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -3 & 9 \end{pmatrix}$$

$$P(A) = \begin{pmatrix} -2 & 14 \\ -8 & 14 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ -3 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 20 \\ -11 & 24 \end{pmatrix}$$

A successive matrix is defined if it is a square matrix

Diagonal and Triangular Matrices: A square matrix $D = [d_{ij}]$ is diagonal if its non-diagonal entries all are zero.

i.e. $d_{ij} = 0$ if $i \neq j$
and $d_{ij} \neq 0$ if $i = j$

Such matrix is denoted by $D = diag(d_{11}, d_{22}, d_{33}, \dots \dots d_{nn})$

For example, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is a diagonal matrix which can be represented by $diag(2,7,4)$

A square matrix A=$[a_{ij}]$ is called *upper triangular* or *triangular* if all entries below the diagonal is equal to zero. That is if a_{ij} = 0 for $i > j$

For example, $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$

Theorem: *Let A= $[a_{ij}]$, B=$[b_{ij}]$are n upper triangular matrix & k is any scalar. Then-*

1. *$A + B, kA, AB$ are triangular matrices with respective diagonals-*
   $(a_{11} + b_{11}, a_{12} + b_{12}, \dots \dots a_{nn} + b_{nn})$, ($ka_{11}$+$ka_{22}$+… … +$ka_{nn}$),
   $(a_{11}b_{11}, a_{12}b_{12}, \dots \dots a_{nn}b_{nn})$

2. *For any polynomial f(x) the matrix f(A) is triangular with diagonal*
   $\big(f(a_{11}), f(a_{12}), \dots \dots f(a_{nn})\big)$

Symmetric Matrices: A square matrix *A* is called symmetric if $A^T = A$.

Equivalently if $A = [a_{ij}]$ then each $a_{ij} = a_{ji}$

Example: $A = \begin{pmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & 8 \end{pmatrix}$

Skew-Symmetric Matrices: A square matrix A is called skew-symmetric if $A^T = A$.

Equivalently if $A = [a_{ij}]$ then each $a_{ij} = -a_{ji}$

Example: $A = \begin{pmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{pmatrix}$

Theorem: *If A and B are symmetric matrix then-*

1. A^T*, A+B, A-B is symmetric*
2. *kA is symmetric for any scalar k*
3. *AB is symmetric if AB = BA*

A matrix A must be square if $A^T = A$ *or* $A^T = -A$

Orthogonal Matrices: A matrix *A* is orthogonal if $A^T = A^{-1}$ i.e. $AA^T = I$
Thus *A* must be square & Invertible.

Theorem: *If A be a real matrix then the following are equivalent-*

1. *A is Orthogonal*
2. *The rows of A from orthogonal set*
3. *The columns of A from orthogonal set*

Normal Matrix: A real matrix *A* is normal if it commutes with transpose.

That is $AA^T = A^T A$

Example: Let, $A = \begin{pmatrix} 6 & -3 \\ 3 & 6 \end{pmatrix}$ then-

$$AA^T = \begin{pmatrix} 6 & -3 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 6 & 3 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 45 & 0 \\ 0 & 45 \end{pmatrix} \quad \& \quad A^T A = \begin{pmatrix} 6 & 3 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 45 & 0 \\ 0 & 45 \end{pmatrix}$$

Every inverse of a matrix is unique

Let, *A* be a matrix. *B* & *C* are two inverse matrix of *A*. Now we have to show that *B=C*

As *B* is an inverse matrix of *A*. So $AB = BA = I$

Again, C is an inverse matrix of A. So $AC = CA = I$

Now, $B = BI = B(AC) = (BA).C = I.C = C$

As we get $B = C$, so every inverse of a matrix is unique.

Theorem: *If A and B be two invertible matrices of same order, then-*

1. $A \text{ is invertible}$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(AB)(B^{-1}A^{-1}) = I$

Theorem: *If the order of the matrices are such that the started operation can be performed then-*

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(kA^T) = k\,A^T$
4. $(AB)^T = B^T.\,A^T$ [Its not $A^T.B^T$]
5. $(A^{-1})^T = (A^T)^{-1}$

If A is symmetric, then A^T is symmetric

Since A is symmetric, we have, $A^T = A$

Now, $(A^T)^T = A = A^T$

$\therefore$ A^T is symmetric

If A is symmetric, then kA is symmetric

Since A is symmetric, we have, $A^T = A$

Now, $(kA)^T = k.A^T = kA$

$\therefore$ kA is symmetric

If A & B are symmetric, then A ± B is symmetric

Since A and B are symmetric, we have, $A^T = A$ and $B^T = B$

Now, $(A \pm B)^T$= A^T ± B^T = A ± B

$\therefore$ A ± B is symmetric

If A & B are symmetric, then AB is symmetric

Since A and B are symmetric, we have, $A^T = A$ and $B^T = B$

Now, $(AB)^T = B^T A^T = BA = AB$

[As A & B are symmetric, so AB = BA]

$\therefore$ AB is symmetric

If A is invertible symmetric, then $\mathbf{A^{-1}}$ is symmetric

Since A is invertible symmetric, we have, $(A^T)^{-1} = A^{-1}$

Now, $(A^{-1})^T = (A^T)^{-1}$

$\therefore\ A^{-1}$ is symmetric

$(\boldsymbol{AB})^{-1} = \boldsymbol{B^{-1}A^{-1}}$

If we can show that, $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$ then we will have simultaneously shown that the matrix AB is invertible and that is

$$(AB)^{-1} = B^{-1}A^{-1}$$

Now, $(AB)(B^{-1}A^{-1}) = A\,(B.B^{-1})\,A^{-1} = A.I.A^{-1} = A.A^{-1} = I$

Again, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}.I.B = B^{-1}.B = I$

A similar argument is $(AB)^{-1} = B^{-1}A^{-1}$

Elementary Matrix: A matrix *A* is called an elementary matrix if it is obtained from the identity matrix of same order *A* by performing a single elementary row operation.

Example: If *I* is an identity matrix, where $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then A matrix-

$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an elementary matrix because

we get *A* by multiplying the 1st row of *I* by 3

Every elementary matrix is invertible and the inverse is also an elementary matrix

If E is an elementary matrix then E results from performing some row operation on I. Let E_0 be the matrix that results when the inverse of this operation is performing on I.

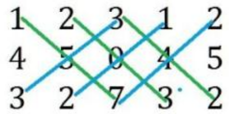
We know, the inverse row operation cancel the effect of each other. It follows that-

$EE_0 = I$ and $E_0E = I$

Thus the elementary matrix E_0 is the inverse of E

Sarrus Diagram: Sarrus' rule or Sarrus' scheme is a method and a memoriz-ation scheme to compute the determinant of a 3×3 matrix.

Exapmle1: Find the value of the determinate by using sarrus diagram $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 7 \end{vmatrix}$

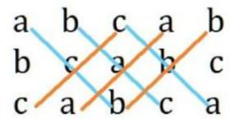


$$= (1.5.7) + (2.0.3) + (3.4.2) - (3.5.3) - (2.0.1) - (7.4.2)$$

$$= -42$$

$$\text{Thus} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 7 \end{vmatrix} = -42$$

Example 2: Appling sarrus diagram show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$



$$= (a.c.b) + (b.a.c) + (c.a.b) - (c.c.c.) - (a.a.a) - (b.b.b)$$

$$= 3abc - a^3 - b^3 - c^3$$

If A is invertible, then AA^T & A^TA are invertible

Since A is invertible, we have, $A.A^{-1} = A^{-1}.A = I$

Now, $(AA^T).(AA^T)^{-1} = (AA^T).[(A^T)^{-1}.A^{-1}]$

$$= A.[A^T.(A^T)^{-1}].A^{-1}$$
$$= A.I.A^{-1}$$
$$= A.A^{-1}$$
$$= I$$

$$(AA^T)^{-1}.(AA^T) = [(A^T)^{-1}.A^{-1}].(A.A^T)$$
$$= (A^T)^{-1}[A^{-1}.A]\,A^T$$
$$= (A^T)^{-1}\,I.A^T$$
$$= (A^T)^{-1}A^T$$
$$= I$$

$\therefore A\,A^T$ is invertible

$$\text{Again, } (A^T A).(A^T A)^{-1} = (A^T A).[A^{-1}.(A^T)^{-1}]$$
$$= A^T [A.A^{-1}] (A^T)^{-1}$$
$$= A^T.I.(A^T)^{-1}$$
$$= A^T.(A^T)^{-1}$$
$$= I$$

$$(A^T A)^{-1}.(A^T A) = [A^{-1}.(A^T)^{-1}].(A^T.A)$$
$$= A^{-1}[(A^T)^{-1} A^T] A$$
$$= A^{-1}.\ I.A$$
$$= A^{-1}.\ A$$
$$= I$$

$\therefore$ $A^T.A$ is invertible

Every square matrix can be written as the sum of a symmetric and skew symmetric matrix

Let, A be a square matrix of order n.

We can write, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$\text{Now, } \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}[(A + A^T)]^T$$
$$= \frac{1}{2}(A^T + A)$$
$$= \frac{1}{2}(A + A^T)$$

$\therefore$ $\frac{1}{2}(A + A^T)$ is symmetric

$$\text{Again, } \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}[(A - A^T)]^T$$
$$= \frac{1}{2}(A^T - A)$$
$$= -\frac{1}{2}(A - A^T)$$

$\therefore$ $\frac{1}{2}(A + A^T)$ is symmetric

Therefore, every square matrix can be written as the sum of a symmetric and skew symmetric matrix

Hermitian & Skew-Hermitian Matrix: Let, A is a complex matrix. Its conjugate transpose is A^H. If $A^H = A$, then A is called hermitian matrix. If $A^H = -A$, then A is called skew-hermitian matrix.

Unitary: Let, A is a complex matrix. Its conjugate transpose is A^H. If $A^H = A$, then A is called unitary if $A^H . A^{-1} = A^{-1} . A^H = I$, i.e. $A^H = A$

Thus A is necessarily be square and invertible.

Normal: Let, A is a square matrix and its conjugate transpose is A^H. Now if $A . A^H$=$A^H . A$, then A is called normal.

Ques: Find A^H , where $A = \begin{pmatrix} 3-5i & 2+4i \\ 6+7i & 1+8i \end{pmatrix}$

We know, $A^H = A^{-T}$

$$\therefore \ A^H = \begin{pmatrix} 3+5i & 6-7i \\ 2-4i & 1-8i \end{pmatrix}$$

System of Linear Equation: We define a *linear equation* of n variables $x_1, x_2, x_3, \dots, x_n$ to be one that can be expressed by $a_1x_1, a_2x_2, a_3x_3, \dots, a_nx_n = b$ where $a_1, a_2, a_3, \dots, a_n$ and b are real constants. The variables in a linear equation are sometimes called *unknowns*

The equation $x + 3y = 7$, $y = \frac{x}{2} + z + 3$ or $x_1 - 2x_2 + 6x_3 = -6$ are linear, but $x + 3\sqrt{y} = 5$, $x + 2y - 5xz = 4$, $y = \sin x$ are not linear.

The *solution* of a linear equation $a_1x_1, a_2x_2, a_3x_3, \dots, a_nx_n = b$ is a sequence of *n* numbers $s_1, s_2, s_3, \dots, s_n$ such that the equation is satisfied when we substitute $x_1 = s_1, x_2 = s_2, x_3 = s_3, \dots, x_n = s_n$ The set of all solutions of the equation is called its *solution* set or sometimes the *general solution* of the equation.

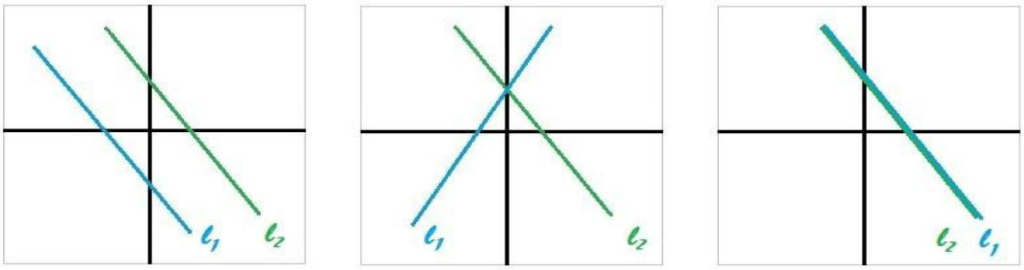
A finite set of linear equations in the variables $x_1, x_2, x_3, \dots, x_n$ is called a *system of linear equations* or a *linear system*. A sequence of numbers $s_1, s_2, s_3, \dots, s_n$ is called a *solution* of the system if $x_1 = s_1, x_2 = s_2, x_3 = s_3, \dots, x_n = s_n$ is a solution of every equation in the system.

A system of equations that has no solutions is said to be *inconsistent*; if there is at least one solution of the system, it is called *consistent*. To illustrate the possibilities that can occur in solving systems of linear equations, consider a general system of two linear equations in the unknowns *x* and *y*:

$$a_1x + b_1y = c_1 \quad [a_1,\ b_1\ are\ not\ both\ zero]$$

$$a_2x + b_2y = c_2 \quad [a_2,\ b_2\ are\ not\ both\ zero]$$

The graphs of these equations are lines; call them l_1 and l_2. Since a point (x, y) lies on a line if and only if the numbers x and y satisfy the equation of the line, the solutions of the system of equations correspond to points of intersection of l_1 and l_2. There are three possibilities, illustrated in figure-



$No\ solution$ $\qquad$ $One\ Solution$ $\qquad$ $Infinitely\ many\ Solutions$

In the 1st figure, the lines l_1 and l_2 may be parallel, in which case there is no intersection and consequently no solution to the system.

In the 2nd figure, the lines l_1 and l_2 may intersect at only one point, in which case the system has exactly one solution.

In the 3rd figure, the lines l_1 and l_2 may coincide, in which case there are infinitely many points of intersection and consequently infinitely many solutions to the system.

Every system of linear equations has no solutions or has exactly one solution, or has infinitely many solutions

Every system of linear equations has no solutions or has exactly one solution, or has infinitely many solutions

If $A\bar{x} = \bar{b}$ is a system of linear equation then one of the following is true-

1. The system has no solution
2. The system has exactly one solution
3. The system has more than one solution

The proof will be complete if we can show that the system has infinitely many solution in case 3.

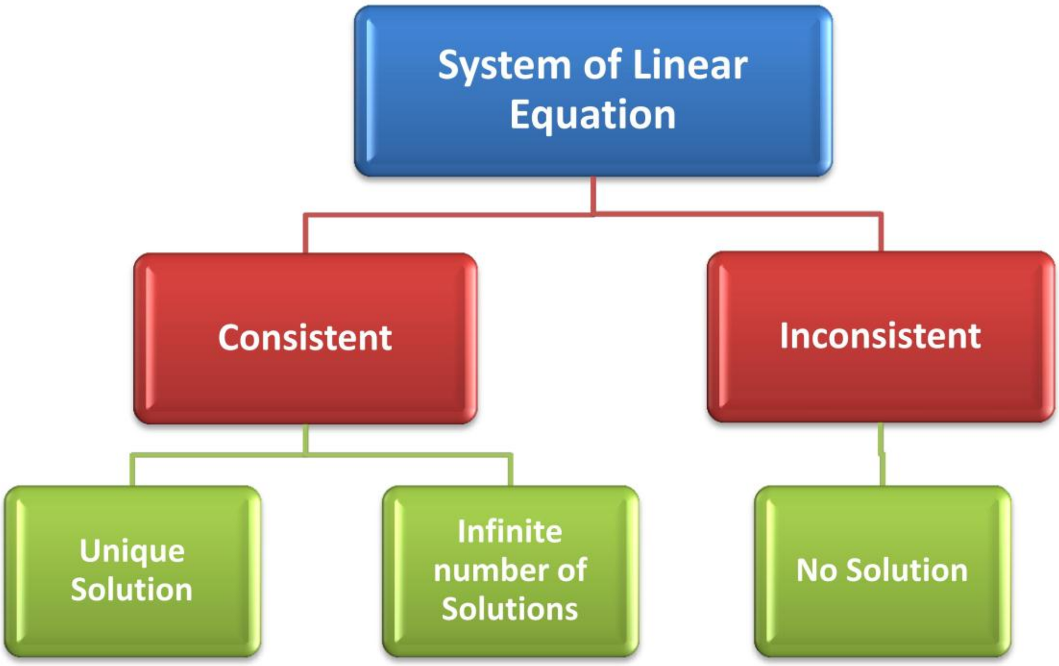
Assume that $A\bar{x} = \bar{b}$ has more than one solution, and let $\overline{x_0} = \overline{x_1} - \overline{x_2}$, where $\overline{x_1}$ and $\overline{x_2}$ are any two distinct solutions. Because $\overline{x_1}$ and $\overline{x_2}$ are distinct, the matrix is nonzero. Moreover

$$A\overline{x_0} = A(\overline{x_1} - \overline{x_2}) = A\overline{x_1} - A\overline{x_2} = \overline{b} - \overline{b} = 0$$

If we now let *k* be any scalar, then

$$A(\overline{x_1} + k\overline{x_0}) = A\overline{x_1} + A(k\overline{x_0}) = \bar{b} + k(A\overline{x_0}) = \bar{b}$$

But this says that $(\overline{x_1} + k\overline{x_0})$ is a solution of $A\bar{x} = \bar{b}$. Since $\overline{x_0}$ is nonzero and there are infinitely many choices for k. So the system has infinitely many solutions.



The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but is easier to solve. This new system is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically-

1. *Multiply an equation through by a nonzero constant.*

2. *Interchange two equations.*

3. *Add a multiple of one equation to another.*

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. *Multiply a row through by a nonzero constant.*

2. *Interchange two rows.*

3. *Add a multiple of one row to another row*

These are called *elementary row operations*. The following example illustrates how these operations can be used to solve systems of linear equations. Since a systematic procedure for finding solutions will be derived in the next section, it is not necessary to worry about how the steps in this example were selected. The main effort at this time should be devoted to understanding the computations and the discussion.

Example: Solve the system of linear equation-

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Solution: To solve this system replace the second equation by (-2) times the first equation added to the second. This yields the system-

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add -3 times the first equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Add -3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ -\frac{1}{2}z &= -\frac{3}{2} \end{aligned}$$

Multiply the third equation by -2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned}$$

Add -1 times the second equation to the first to obtain

$$\begin{aligned} x \quad + \frac{11}{2}z &= \frac{35}{2} \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ *times the* 3^{rd} *row to the* 1^{st} *&* $\frac{7}{2}$ *times the* 3^{rd} *row to the* 2^{nd} *to obtain*

$$\begin{array}{llll} x & & & = 1 \\ & y & & = 2 \\ & & z & = 3 \end{array}$$

$\therefore$ The solution is $x = 1,\ y = 2,\ z = 3$ is now evident.

NB: Here, we solve the system of linear equations by operating on the equations in the system. This process is called *back substitution*.

Gauss Elimination: If we solve the same system by operating on the rows of the augmented matrix then its called Gauss Elimination. Gauss elimination is a systematic procedure to simplify an augmented matrix to a reduced form.

For example the linear system, above can be written as-

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second row by $\frac{1}{2}$ *to obtain*

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add −3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third row by −2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add −1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ *times the* 3^{rd} *row to the* 1^{st} *&* $\frac{7}{2}$ *times the* 3^{rd} *row to the* 2^{nd} *to obtain*

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

∴ The solution is $x = 1,\ y = 2,\ z = 3$ is now evident.

In this example, we solved a linear system in the unknowns x, y, and z by reducing the augmented matrix to the form-

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

For which the solution is $x = 1,\ y = 2,\ z = 3$ became evident. This is an example of a matrix that is in ***reduced row-echelon form***. To be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first non-zero number in the row is a 1. We call this a *leading 1*.

2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row-echelon form***. (Thus, a matrix in reduced row-echelon form is of necessity in row-echelon form, but not conversely.

❖ **What's the difference between row echelon form and reduced row echelon form?**

An augmented matrix is in echelon form if-

1. *Each nonzero row lies above every zero row.*
2. *The leading entry of a nonzero row lies in a column to the right of the column with the leading entry of any preceding row.*
3. *If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.*

An reduced augmented matrix is in echelon form if-

1. *Each nonzero row lies above every zero row.*
2. *The leading entry of a nonzero row lies in a column to the right of the column with the leading entry of any preceding row.*
3. *If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.*
4. *If a column contains the leading entry of some row, then all the other entries of that column are 0.*
5. *The leading entry of each nonzero row is 1.*

Here are some augmented matrix which are not reduced row-echelon form-

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 8 & 0 \\ 0 & 0 & 1 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Here are some augmented matrix which are not reduced row-echelon form but echelon

$$\left(\begin{array}{ccccc|c} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 2 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Here are some augmented matrix which are not row-echelon form-

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 1 & 0 & 7 \end{array}\right), \left(\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Rank of a Matrix: The number of non-zero rows in echelon form of a matrix is called the rank of that matrix.

> *If Rank* $[A{:}\,H]$ *=Rank* $[A] = n$ *then the system has a unique solution*
> *If Rank* $[A{:}\,H]$ *=Rank* $[A] < n$ *then the system has many solutions*

Cramer's Rule: If $Ax = b$ is a system of n linear equation, with n unknowns, such that $det(A) \neq 0$ then the system has a unique solution.

The solution is-

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \text{where } i = 1, 2, 3, \dots, n$$

where A_i is the matrix formed by replacing the *i* th column of A by the column vector b.

For example,

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

then x, y and z can be found from

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Example 1: Solve the system of linear equation

$$\begin{aligned} x + \ y + 2z &= 8 \\ -x - 2y + 3z &= 1 \\ 3x - 7y + 4z &= 10 \end{aligned}$$

Solution: Hence the augmented matrix for the system of linear equation is-

$$\text{Let, A} = \begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \qquad [R_2{}' = R_1 + R_2 \quad \& \quad R_3{}' = R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \qquad [R_2{}' = (-1)\, R_2]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \qquad [R_3{}' = R_3 + 10\, R_2]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad [R_3{}' = (-\tfrac{1}{52})\, R_3]$$

This is an row- echelon form of [A : H]

Here, Rank [A] = 3, which is equal to unknowns. So the system of linear equation has a unique solution.

Now,
$$x + y + 2z = 8 \ldots\ldots (i)$$
$$y - 5z = -9 \ldots\ldots (ii)$$
$$z = 2 \ldots\ldots (iii)$$

From (ii) $\quad y - 10 = -9$

$\therefore y = 1$

From (i) $\quad x + 1 + 4 = 8$

$\therefore x = 3$

Solution of the system $(x, y, z) = (3,1,2)$

Example 2: Solve the system of linear equation

$$2x + 2y + 2z = 0$$
$$-2x + 5y + 2z = 1$$
$$8x + y + 4z = -1$$

Solution: Hence the augmented matrix for the system of linear equation is-

$$\text{Let, } A = \begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \qquad [R_1{}' = \left(\tfrac{1}{2}\right) R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & -7 & -4 & -1 \end{bmatrix} \qquad [R_2{}' = R_2 + 2\, R_1 \quad \& \quad R_3{}' = R_3 - 8\, R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left[R_2' = (\tfrac{1}{7})\, R_2 \quad \& \quad R_3' = R_3 + R_2\right]$$

This is an row- echelon form. Here, Rank [A : H] = Rank [A] = 2 < Unknowns.

So the system of linear equation has infinitely many solutions.

Now, $x + y + z = 0 \ldots\ldots(i)$

$y + \frac{4}{7}z = \frac{1}{7} \ldots\ldots(ii)$

Here z is a free variable

Let, $z = a$

Then from (ii) we get, $y = -\frac{4a-1}{7}$

from (iii) we get $x = -\frac{3a+1}{7}$

Solution of the system $(x, y, z) = \left(-\frac{3a+1}{7},\ -\frac{4a-1}{7},\ a\right)$

Example 3: Solve the system of linear equation

$$\begin{aligned} -2y + 3z &= 1 \\ 3x + 6y - 3z &= -2 \\ 6x + 6y + 3z &= 5 \end{aligned}$$

Solution: Hence the augmented matrix for the system of linear equation is-

$$\text{Let, } A = \begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \quad \left[R_1' = R_2 \quad \& \quad R_2' = R_1\right]$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \quad \left[R_1' = \tfrac{1}{3}\, R_1\right]$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 6 & 3 & 5 \end{bmatrix} \quad \left[R_2' = (-\tfrac{1}{2})\, R_2\right]$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix} \qquad [R_3{}' = R_3 - 6R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix} \qquad [R_3{}' = R_3 + 6R_2]$$

This is an row- echelon form. Here, Rank [A : H] ≠ Rank [A]

So the system of linear equation has no solution.

Example 4: Solve the system of linear equation

$$x + y + 2z = 9$$
$$2x + 4y - 3z = 1$$
$$3x + 6y - 5z = 0$$

Solution: Hence the augmented matrix for the system of linear equation is-

$$\text{Let, } A = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix} \qquad [R_2{}' = R_2 - 2R_1 \quad \& \quad R_3{}' = R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix} \qquad [R_2{}' = \left(\tfrac{1}{2}\right) R_2]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \qquad [R_3{}' = R_3 - 3\,R_2]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad [R_3{}' = (-2)\,R_3]$$

This is an row- echelon form of [A : H]

Here, Rank [A : H] = Rank [A] = 3, which is equal to unknowns. So the system of linear equation has a unique solution.

Now, $x + y + 2z = \;\; 9 \ldots\ldots(i)$

$$y-\frac{7}{2}z=-\frac{17}{2}\ldots\ldots(ii)$$
$$z=\quad 3\ldots\ldots(iii)$$

From (ii) $\quad y-\frac{21}{2}=-\frac{17}{2}$

$$\therefore y=2$$

From (i) $\quad x+2+6=9$

$$\therefore x=1$$

Solution of the system $(x,y,z)=(1,2,3)$

Example 5: Solve the system of linear equation

$$x+2y-3z+w=0$$
$$x-3y+z-2w=0$$
$$2x+y-3z+5w=0$$

Solution: Since the number of unknowns are more than the number of equations so the system has infinitely many solutions.

Hence the augmented matrix for the system of linear equation is-

Let, $A=\begin{bmatrix}1 & 2 & -3 & 1\\ 1 & -3 & 1 & -2\\ 2 & 1 & -3 & 5\end{bmatrix}$

$\sim \begin{bmatrix}1 & 2 & -3 & 1\\ 0 & -5 & 4 & -3\\ 0 & -3 & 3 & 3\end{bmatrix}$ $\quad [R_2{}' = R_2 - R_1 \text{ \& } R_3{}' = R_3 - 2R_1]$

$\sim \begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & -\frac{4}{5} & \frac{3}{5}\\ 0 & -3 & 3 & 3\end{bmatrix}$ $\quad [R_2{}' = -\frac{1}{5}R_2]$

$\sim \begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & -\frac{4}{5} & \frac{3}{5}\\ 0 & 0 & \frac{3}{5} & \frac{24}{5}\end{bmatrix}$ $\quad [R_3{}' = R_3 + 3R_2]$

$\sim \begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & -\frac{4}{5} & \frac{3}{5}\\ 0 & 0 & 1 & 8\end{bmatrix}$ $\quad [R_3{}' = \frac{5}{3}R_3]$

This is an echelon form.

Consider the system of linear equation-

$$x + 2y - 3z + w = 0 \quad \text{.............(i)}$$
$$y - \frac{4}{5}z + \frac{3}{5}w = 0 \quad \text{...........(ii)}$$
$$z + 8w = 0 \quad \text{.............(iii)}$$

Here w is the free variable for this system.

Let, $w = a$

Then from (iii) we get $z = -8a$

from (ii) we get $y = -7a$

from (i) we get $x = -11a$

Hence the solution of the system is $(x, y, z, w) = (-11a, -7a, -8a,\ a)$

Practice problem:

1. For which value of a in the following system of linear equation has-
   $(i)\ no\ solution\ (ii)\ unique\ solution\ (iii)\ infinitely\ many\ solutions$
   $x + y - z = 1,\ \ 2x + 3y + az = 3,\ \ x + ay + 3z = 2$
   $[\,Ans{:}\ (i)\ a = -3,\ (ii)\ a \neq -3\ or\ a \neq 2\ \ (iii)\ a = 2\,]$

2. Find the solution of the system of linear equation-
   $x + 2y - z = 0,\ \ 2x + 5y + 2z = 0,\ \ x + 4y + 7z = 0,\ \ x + 3y + 3z = 0$

3. Find the solution of the system of linear equation-
   $x + 2y - 3z = 0,\ \ 2x + 5y + 2z = 0,\ \ 3x - y - 4z = 0$

Homogeneous System of Linear Equation: A system of linear equations is said to be homogeneous if the constant terms are all zero; that is, the system has the form-

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

Every homogeneous system of linear equations is consistent, since all such systems have $x_1 = 0, x_2 = 0, x_3 = 0, \dots, x_n = 0$ as a solution. This solution is called the *trivial solution*; if there are other solutions, they are called *nontrivial solutions.*

Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

1. The system has only the trivial solution.
2. The system has infinitely many solutions in addition to the trivial solution.

A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

Vector Space: Let V be a non-empty set of vectors under two operation of vector addition and scalar multiplication. Then V is called vector space and it satisfies the following axioms-

Theorem: *For all* $\vec{u}$, $\vec{v}$, $\vec{w} \in V$ *and scalars k, l-*

1. $\vec{u} + \vec{v} \in V$
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
4. *There is* $\vec{0} \in V$*, such that* $\vec{u} + \vec{0} = \vec{u}$ *, here 0 is called zero/null vector.*
5. *For each* $\vec{u} \in V$*, there is a* $-\vec{u}$ *, such that* $\vec{u} + (-\vec{u}) = \vec{0}$
6. $k\vec{u} \in V$
7. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$
8. $(k + l)\vec{u} = k\vec{u} + l\vec{u}$
9. $(kl)\vec{u} = k(l\vec{u})$
10. $1\vec{u} = \vec{u}$

Sub Space: A subset W of a vector space V is called a subspace of V if W itself a vector space under the operations of V.

Every vector spaces V itself and the set of only zero vectors $\{\vec{0}\}$ are subspace of the vector space. These two subspaces are called improper subspace.

$$\mathbf{0}.\vec{\boldsymbol{u}} = \vec{\mathbf{0}}$$

We can write ,

$$0.\vec{u} + 0.\vec{u} = (0+0)\vec{u}$$
$$\text{Or, } 0.\vec{u} + 0.\vec{u} = 0\vec{u}$$
$$\text{Or, } 0.\vec{u} + 0.\vec{u} + (-0\vec{u}) = 0\vec{u} + (-0\vec{u})$$
$$\text{Or, } 0.\vec{u} + \vec{0} = \vec{0}$$
$$\therefore\ 0.\vec{u} = \vec{0}$$

$$(-1)\vec{u} = -\vec{u}$$

To show $(-1)\vec{u} = -\vec{u}$ we must demonstrate that $\vec{u} + (-1)\vec{u} = \vec{0}$

$$\vec{u} + (-1)\vec{u} = 1\vec{u} + (-1)\vec{u}$$
$$= [1+(-1)]\vec{u}$$
$$= 0.\vec{u}$$
$$= \vec{0}$$
$$\therefore (-1)\vec{u} = -\vec{u}$$

If $\vec{u}, \vec{v} \in V$ then prove that $k(\vec{u} - \vec{v}) = k\vec{u} - k\vec{v}$

We know that, $\vec{u} - \vec{v} = \vec{u} + (\overline{-\vec{v}})$

And $k(-\vec{v}) = -k\vec{v}$

Now, $k(\vec{u} - \vec{v}) = k[\vec{u} + (-\vec{v})]$

$= k\vec{u} + k(-\vec{v})$

$= k\vec{u} - k\vec{v}$

Linearly Dependent: A subset of a vector space is linearly independent if none of its elements is a linear combination of the others. Otherwise it is linearly dependent.

Linear Combination: Let V *be a* vector space over a field k. A vector $\vec{v} \in V$ is called a linear combination of a set S=$\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \dots, \vec{v_n}\}$ of vectors in $\vec{v}$ if there are scalars $c_1, c_2, c_3, c_4, \dots, c_n$ such that $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} + \cdots + c_n\vec{v_n}$

Let, $V = \mathbb{R}^3$. Then any $\vec{v} \in V$ is the linear combination of S=$\{e_1, e_2, e_3\}$ when $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

Then, $\vec{v} = (a, b, c) = ae_1 + be_2 + ce_3$

Span: Let V *be a* vector space over a field k. A subset S=$\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \dots, \vec{v_n}\} \subseteq V$ is called span of V if S is linearly independent. And every $\vec{v} \in V$ is the linear combination of S.

Let, $V = \mathbb{R}^3$ Span V S=$\{e_1, e_2, e_3\}$

when $(1,0,0), (0,1,0), (0,0,1)$

$a(1,0,0) + b(0,1,0) + c(0,0,1)$

$\therefore (a, b, c) = (0, 0, 0)$

Basis: Suppose S=$\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}, \ldots, \overrightarrow{v_n}\}$ is a set of vectors from the vector space V. Then S is called a basis (plural is bases) for V if both of the following conditions hold.

(a) $Span\ S = V$, *i.e.* S spans the vector space V.

(b) S is a linearly independent set of vectors.

Example 1: Determine whether or not each of the following form a basis of $\mathbb{R}^3$

(a) $(1,1,1)(1,0,1)$

(b) $(1,1,1)(1,2,3)(2,-1,1)$

(c) $(1,2,3)(1,3,5)(1,0,1)(2,3,0)$

(d) $(1,1,2)(1,2,5)(5,3,4)$

Solution:

(a) Since a basis of $\mathbb{R}^3$ must contain exactly 3 elements because $\dim \mathbb{R}^3 = 3$
So its not a basis of $\mathbb{R}^3$

(b) The three vectors form a basis if and only if they are linearly independent. Thus from the matrix, whose roes are the given vectors and row reduce the matrix to echelon from-

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

(c) Since a basis of $\mathbb{R}^3$ must contain exactly 3 elements because $\dim \mathbb{R}^3 = 3$
So its not a basis of $\mathbb{R}^3$

(d) The three vectors form a basis if and only if they are linearly independent. Thus from the matrix, whose roes are the given vectors and row reduce the matrix to echelon from-

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 7 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row, hence the three vectors are linearly dependent. So they do not form a basis of $\mathbb{R}^3$

Example 2: Determine whether $(1,1,1,1), (1,2,3,2), (2,5,6,4), (2,6,8,5)$ form a basis of $\mathbb{R}^4$. If not , find the dimension of the subspace they span.

Solution: From the matrix whose rows are the given vectors and row reduce to echelon form-

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row, hence the four vectors are linearly dependent. So they do not form a basis of $\mathbb{R}^4$. Since the echelon matrix has three nonzero rows, the four vectors span a subspace of dimension 3.

Example 3: Let w be a subspace on $\mathbb{R}$, generated by the vectors $(1, -2,\ 5, -3)$, $(2,\ 3, 1, -4)$, $(3, 8, -3, -5)$. Find the dimension of w. **Extend the basis of w to a basis of $\mathbb{R}^4$**

Solution: From the matrix whose are the given vectors and row reduced to echelon form-

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row, hence the three vectors are linearly dependent. So they do not form a basis of $\mathbb{R}^4$. Since the echelon matrix has two nonzero rows, the three vectors span a subspace of dimension 2.

Inner Product: An inner product on a real vector space V is a function that associates a real number $\langle \vec{u}, \vec{v} \rangle$ with each pair of vectors $\vec{u}$ and $\vec{v}$ in V in such a way that the following axioms are satisfied for all vectors $\vec{u}$, $\vec{v}$ & $\vec{z}$ in V and all scalars k-

Properties:

1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ [Symmetric]
2) $\langle \vec{u} + \vec{v},\ \vec{z} \rangle = \langle \vec{u}, \vec{z} \rangle + \langle \vec{v}, \vec{z} \rangle$ [Additivity]
3) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$ [Homogeneity]
4) $\langle \vec{v}, \vec{v} \rangle \geq 0$ [Positivity]
5) $\langle \vec{u}, \vec{0} \rangle = \langle \vec{0}, \vec{u} \rangle = 0$
6) $\langle \vec{u} + \vec{v},\ \vec{z} \rangle = \langle \vec{u}, \vec{z} \rangle + \langle \vec{v}, \vec{z} \rangle$
7) $\langle \vec{u} - \vec{v},\ \vec{z} \rangle = \langle \vec{u}, \vec{z} \rangle - \langle \vec{v}, \vec{z} \rangle$
8) $\langle \vec{u},\ \vec{v} + \vec{z} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{z} \rangle$
9) $\langle \vec{u},\ \vec{v} - \vec{z} \rangle = \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{z} \rangle$

$\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = 0$

Inner Product Space: A real vector space with an inner product is called a real inner product space.

If $\vec{u} = (u_1,\ u_2,\ u_3,\ \dots, u_n)$ and $\vec{v} = (v_1,\ v_2,\ v_3,\ \dots, v_n)$ are vectors in $\mathbb{R}^n$, then the formula $\langle \vec{u}, \vec{v} \rangle = \vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n$ defines $\langle \vec{u}, \vec{v} \rangle$ to be the Euclidean inner product on $\mathbb{R}^n$.

Weighted Euclidean Inner Product: Let $\vec{u} = (u_1,\ u_2)$and $\vec{v} = (v_1,\ v_2)$ be vectors in $\mathbb{R}^2$. Verify that the weighted Euclidean inner product $\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = 3u_1v_1 + 2u_2v_2$ satisfies the four inner product axioms.

Norm: If V is an inner product space, then the norm (or length) of a vector $\vec{u}$ in V is denoted by $\|\vec{u}\|$ and is defined by $\|\vec{u}\| = \langle \vec{u}, \vec{u} \rangle^{\frac{1}{2}}$.

If a vector has norm 1, then we say that it is a unit vector.

Distance: The distance between two points (vectors) $\vec{u}$ and $\vec{v}$ is denoted by $d(\vec{u}, \vec{v})$ and is defined by $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$.

Example: If $\vec{u} = (u_1,\ u_2,\ u_3,\ \dots, u_n)$ and $\vec{v} = (v_1,\ v_2,\ v_3,\ \dots, v_n)$ are vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|\vec{u}\| = \langle \vec{u}, \vec{u} \rangle^{\frac{1}{2}} = (\vec{u}, \vec{u})^{\frac{1}{2}} = \sqrt{u_1^2 + u_2^2 + u_3^2 + \cdots u_n^2}$$

And $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \langle \vec{u} - \vec{v},\ \vec{u} - \vec{v} \rangle^{\frac{1}{2}}$

$$= [(\vec{u} - \vec{v}).(\vec{u} - \vec{v})]^{\frac{1}{2}}$$
$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 + \cdots + (u_n - v_n)^2}$$

$$\langle \vec{\boldsymbol{u}},\ \vec{\boldsymbol{v}} \rangle = \langle \vec{\boldsymbol{v}},\ \vec{\boldsymbol{u}} \rangle$$

We know that, $\langle \vec{u},\ \vec{v} \rangle = \langle (u_1, u_2),\ (v_1, v_2) \rangle$

$$= 3u_1v_1 + 2u_2v_2$$
$$= 3v_1u_1 + 2v_2u_2$$
$$= \langle \vec{v},\ \vec{u} \rangle$$

$$\langle \vec{\boldsymbol{u}} + \vec{\boldsymbol{v}},\ \vec{\boldsymbol{z}} \rangle = \langle \vec{\boldsymbol{u}},\ \vec{\boldsymbol{z}} \rangle + \langle \vec{\boldsymbol{v}},\ \vec{\boldsymbol{z}} \rangle$$

We know that, $\langle \vec{u} + \vec{v},\ \vec{z} \rangle = \langle (u_1 + v_1,\ u_2 + v_2)\ ,\ (z_1, z_2) \rangle$

$$= 3(u_1 + v_1)z_1 + 2(u_2 + v_2)z_2$$
$$= (3u_1z_1 + 2u_2z_2) + (3v_1z_1 + 2v_2z_2)$$
$$= \langle \vec{u},\ \vec{z} \rangle + \langle \vec{v},\ \vec{z} \rangle$$

$$\langle \boldsymbol{k}\vec{\boldsymbol{u}},\ \vec{\boldsymbol{v}} \rangle = \boldsymbol{k}\, \langle \vec{\boldsymbol{v}},\ \vec{\boldsymbol{u}} \rangle$$

We know that, $\langle k\vec{u},\ \vec{v} \rangle = \langle (ku_1, ku_2),\ (v_1, v_2) \rangle$

$$= 3k\, u_1v_1 + 2k\, u_2v_2$$
$$= k\,(3v_1u_1 + 2v_2u_2)$$
$$= k\, \langle \vec{v},\ \vec{u} \rangle$$

N.B: $k\langle \vec{u},\ \vec{v} \rangle = \langle k\vec{u},\ \vec{v} \rangle = \langle \vec{u},\ \overrightarrow{kv} \rangle = \langle \vec{v},\ k\vec{u} \rangle = \langle \overrightarrow{kv},\ \vec{u} \rangle$

$$\langle \vec{u},\ \vec{u}\ \rangle \geq 0$$

We know that, $\langle \vec{u},\ \vec{u} \rangle = \langle (u_1, u_2),\ (u_1, u_2) \rangle$

$= 3{u_1}^2 + 2{u_2}^2$

$\therefore \quad \langle \vec{u},\ \vec{u}\ \rangle \geq 0$

Further $\langle \vec{u},\ \vec{u} \rangle = 0$

Or, $3{u_1}^2 + 2{u_2}^2 = 0$

$\therefore \quad u_1 = 0$ and $u_2 = 0$

$$\langle \vec{u} - \vec{v},\ \vec{z} \rangle = \langle \vec{u},\ \vec{z} \rangle - \langle \vec{v},\ \vec{z} \rangle$$

We know that, $\langle \vec{u} - \vec{v},\ \vec{z} \rangle = \langle \vec{u},\ \vec{z} \rangle + (-)\ \langle \vec{v},\ \vec{z} \rangle$

$= \langle \vec{u},\ \vec{z} \rangle - \langle \vec{v},\ \vec{z} \rangle$

$$\langle \vec{u},\ \vec{v} - \vec{z} \rangle = \langle \vec{u},\ \vec{v} \rangle - \langle \vec{u},\ \vec{z} \rangle$$

We know that, $\langle \vec{u},\ \vec{v} - \vec{z} \rangle = \langle \vec{v} - \vec{z},\ \vec{u} \rangle$

$= \langle \vec{v},\ \vec{u} \rangle + (-)\langle \vec{z},\ \vec{u} \rangle$

$= \langle \vec{u},\ \vec{v} \rangle - \langle \vec{u},\ \vec{z} \rangle$

Cauchy-Schwartz's Inequality

Theorem: For any vectors $\vec{u} = \{u_1, u_2, u_3, \dots, u_n\}$ and $\vec{v} = \{v_1, v_2, v_3, \dots, v_n\}$ in $\mathbb{R}^n$

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\ \|\vec{v}\|$$

Prove of Cauchy-Schwartz's Inequality

We have to prove, $|\vec{u}.\vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$

If $\vec{u} = 0$ or $\vec{v} = 0$, then the inequality is true. Thus we have to consider the cases in which $\vec{u} \neq 0$ & $\vec{v} \neq 0$ i.e $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$

Let, $a = \langle \vec{u}, \vec{u} \rangle$, $b = 2\langle \vec{u}, \vec{v} \rangle$, $c = \langle \vec{v}, \vec{v} \rangle$

And t be any real number. By the positive axiom, the inner product of any vector with itself is always non-negative. Therefore-

$$\langle (t\vec{u} + \vec{v}), (t\vec{u} + \vec{v}) \rangle \geq 0$$

Or, $\langle \vec{u}, \vec{u} \rangle t^2 + 2\langle \vec{u}, \vec{v} \rangle t + \langle \vec{v}, \vec{v} \rangle \geq 0$

Or, $at^2 + bt + c \geq 0$

Thus in-equality implies that the quadric polynomial $at^2 + bt + c$ has *either no real roots* or *a repeated roots*. Therefore its discriminant must satisfy the inequality $b^2 - 4ac \leq 0$. Expressing the co-efficients a, b & c in term of the vectors $\vec{u}$ and $\vec{v}$ gives-

$$4\langle \vec{u}, \vec{v} \rangle^2 - 4\langle \vec{u}, \vec{u} \rangle . \langle \vec{v}, \vec{v} \rangle \leq 0$$

Or, $\langle \vec{u}, \vec{v} \rangle^2 \leq \langle \vec{u}, \vec{u} \rangle . \langle \vec{v}, \vec{v} \rangle$

Taking square root on the both sides we get,

Or, $|\langle \vec{u}, \vec{v} \rangle| \leq \langle \vec{u}, \vec{u} \rangle^{\frac{1}{2}} . \langle \vec{v}, \vec{v} \rangle^{\frac{1}{2}}$

Or, $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|$

Hence the theorem is proved.

Theorem: If $\vec{u}$ & $\vec{v}$ are vectors in an inner product space V & k is any scalar, then-

1) $\|\vec{u}\| \geq 0$
2) $\|\vec{u}\| = 0$ if and only if $\vec{u} = 0$
3) $\|k\vec{u}\| = |k|\|\vec{u}\|$
4) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ [Triangle inequality]
5) $d(\vec{u}, \vec{v}) \geq 0$
6) $d(\vec{u}, \vec{v}) = 0$ if and only if $\vec{u} = \vec{v}$
7) $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$
8) $d(\vec{u}, \vec{v}) \leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v})$ [Triangle inequality]

$$\|\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}\| \leq \|\vec{\boldsymbol{u}}\| + \|\vec{\boldsymbol{v}}\|$$

By definition, $\|\vec{u} + \vec{v}\|^2 = \langle \vec{u} + \vec{v} \, , \; \vec{u} + \vec{v} \, \rangle$

$$= \langle \vec{u}, \vec{u} \rangle + 2\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle$$

$$= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2$$

As $ab \leq |ab|$, so we can write, $\langle \vec{u}, \vec{v} \rangle \leq |\langle \vec{u}, \vec{v} \rangle|$

Or, $2\langle \vec{u}, \vec{v} \rangle \leq 2|\langle \vec{u}, \vec{v} \rangle|$

Again from Cauchy-Schwartz in-equality

$$\|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2|\langle \vec{u}, \vec{v} \rangle| + \|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$$

$$= [\|\vec{u}\| + \|\vec{v}\|]^2$$

$$\therefore \quad \|\vec{u} + \vec{v}\|^2 \leq [\|\vec{u}\| + \|\vec{v}\|]^2$$

Or, $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Orthogonal & Orthogonal Set: Two vectors $\vec{u}$ and $\vec{v}$ are said to be orthogonal if $\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = 0$. A set S $= \{\overrightarrow{u_1}, , \overrightarrow{u_2}, \overrightarrow{u_3} \; \dots , \overrightarrow{u_n}\}$ is called orthogonal set if $\langle \overrightarrow{u_\iota}, \overrightarrow{v_\iota} \rangle = 0$ for any $\overrightarrow{u_\iota}, \overrightarrow{v_\iota} \in S$

Example 1: If $S = \{e_1, e_3, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$ is called a orthogonal set, $\|e_1\| = \|e_2\| = \|e_3\| = \sqrt{1^2 + 0^2 + 0^2}$

Here, $\langle e_1, e_2 \rangle = \langle e_1, e_3 \rangle = \langle e_2, e_3 \rangle = 0$ and $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1$

$\therefore$ S is an orthogonal basis of $\mathbb{R}^3$.

More generally the usual basis of $\mathbb{R}^n$is orthogonal for every n

Example 2: $\vec{u} = (1,1,1,0)$, $\vec{v} = (1, \text{k}, -3, 5)$ Find the value of k for which $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^4$ are orthogonal.

Solution: Here, $\langle \vec{u}, \vec{v} \rangle = 1 + \text{k} - 3 + 0 = k - 2$

$\vec{u}$ and $\vec{v}$ in $\mathbb{R}^4$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

$$\therefore \quad k - 2 = 0$$

Or, $k = 2$

Orthonormal & Orthonormal Set: A vector $\vec{u}$ is said to be orthonormal if $\|\vec{u}\| = 1$. A set S is called orthonormal if $\|\vec{u}\| = 1$ for all $\vec{u} \in$ S

Example 3: Find an orthogonal and an orthonormal basis of $\mathbb{R}^3$ corresponding to the basis $B = \{(1,1,1),(0,1,1),(0,0,1)\}$ of $\mathbb{R}^3$

Solution: Let, $\overrightarrow{u_1} = (1,1,1)$, $\overrightarrow{u_2} = (0,1,1)$, $\overrightarrow{u_3} = (0,0,1)$

Let, $\{\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{v_3}\}$ is an orthogonal and $\{\overrightarrow{w_1},\overrightarrow{w_2},\overrightarrow{w_3}\}$ is an orthonormal basis corresponding the basis $\{\overrightarrow{u_1},\overrightarrow{u_2},\overrightarrow{u_3}\}$

Now,

$$\overrightarrow{v_1} = \overrightarrow{u_1} = (1,1,1)$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\langle\overrightarrow{u_2},\overrightarrow{v_1}\rangle}{\|\overrightarrow{v_1}\|^2}\,\overrightarrow{v_1} = (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) \qquad \left[\|\vec{v}\| = \langle\vec{v},\vec{v}\rangle^{\frac{1}{2}}\right]$$

$$\overrightarrow{v_3} = \overrightarrow{u_3} - \frac{\langle\overrightarrow{u_3},\overrightarrow{v_1}\rangle}{\|\overrightarrow{v_1}\|^2}\,\overrightarrow{v_1} - \frac{\langle\overrightarrow{u_3},\overrightarrow{v_2}\rangle}{\|\overrightarrow{v_2}\|^2}\,\overrightarrow{v_2}$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \left(\frac{\frac{1}{3}}{\frac{2}{3}}\right)\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$$

$$= (0,0,1) - \left[\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) + \frac{1}{2}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)\right]$$

$$= (0,0,1) - \left[\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) + \left(-\frac{1}{3},\frac{1}{6},\frac{1}{6}\right)\right]$$

$$= (0,0,1) - \left(0,\frac{1}{2},\frac{1}{2}\right)$$

$$= \left(0,-\frac{1}{2},\frac{1}{2}\right)$$

$\therefore\ \overrightarrow{v_1} = (1,1,1)$, $\overrightarrow{v_2} = \left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$, $\overrightarrow{v_3} = \left(0,-\frac{1}{2},\frac{1}{2}\right)$ form an *orthogonal basis* for $\mathbb{R}^3$

Norm of the vectors-

$$\|\overrightarrow{v_1}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3},$$

$$\|\overrightarrow{v_2}\| = \sqrt{\left(-\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2} = \frac{\sqrt{6}}{3}$$

$$\|\overrightarrow{v_3}\| = \sqrt{(0)^2+\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

So an *orthonormal basis* for $\mathbb{R}^3$ is-

$$\overrightarrow{w_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$$\overrightarrow{w_2} = \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|} = \left(-\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$$

$$\overrightarrow{w_3} = \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|} = \left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$\therefore \ \overrightarrow{w_1} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad \overrightarrow{w_2} = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \quad \overrightarrow{w_3} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

form an *orthonormal basis* for $\mathbb{R}^3$

Orthogonal Complement: If W is a subspace of V, then the orthogonal complement of W is denoted by $W^{\perp}$ and defined by $W^{\perp} = \{\vec{u} \in V | \langle \vec{u}, \vec{w} \rangle = 0 \text{ for all } \vec{w} \in W\}$

Jobayer Al-Mahmud

Department of Mathematics,
Shahjalal University of Science & Technology,
Sylhet, Bangladesh

facebook.com/jamahmud