Combinatorial Analysis

Binomial coefficients identities

The binomial coefficient $\binom{n}{k}$ counts the number of k-subsets of a set of n elements. These numbers have many fascinating properties and satisfy a number of interesting identities. Recall that

$$\binom{n}{k} = \binom{n}{n-k}.$$

Next we prove some more identities. Use combinatorial proofs.

Exercise 1 *Show that, for all n and k positive integers such that $0 \le k \le n$ (except $n = k = 0$)*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

This identity is called Pascal's formula.

Exercise 2 *For $n \ge 1$,*

$$\sum_{k \ge 0} \binom{n}{2k} = 2^{n-1}.$$

Exercise 3 *Show that, for all n and k positive integers such that $0 \le k \le n$,*

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Exercise 4 *For $n \ge 1$,*

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Exercise 5 *Dividing both sides of the last identity by 2^n allows us to give a different combinatorial proof of the equivalent identity:*

$$\frac{\sum_{k=0}^{n} k\binom{n}{k}}{2^n} = \frac{n}{2}.$$

The next identity is called Vandermonde's identity.

Exercise 6 *For $m \ge 0$, $n \ge 0$,*

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

The next identity has an interesting application to number theory.

Exercise 7 *For* $0 \le m \le k \le n$*,*

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

As a simple consequence of this last identity, Erdos and Szekeres proved the following simple fact about binomial coefficients. It seems that this was not known prior to 1978.

Exercise 8 *For* $0 < m \le k < n$*,* $\binom{n}{m}$ *and* $\binom{n}{k}$ *have a nontrivial common factor, that is, gcd* $\left(\binom{n}{m}, \binom{n}{k}\right) > 1$.

Next we examine identities involving the quantity $\left(\binom{n}{k}\right)$, spoken "$n$ multichoose k", which counts the ways to select k objects from a set of n elements, where order is not important, but repetition is allowed.

Exercise 9 $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$.

Exercise 10 *For* $0 \le n \le m$*,*

$$\left(\binom{n}{m-n}\right) = \binom{m-1}{n-1}.$$

Not surprisingly, there are many multichoose identities that resemble earlier binomial identities. We begin with the Pascal-like identity.

Exercise 11 *For* $n \ge 0$*,* $k \ge 0$ *(except* $n = k = 0$*),*

$$\left(\binom{n}{k}\right) = \left(\binom{n}{k-1}\right) + \left(\binom{n-1}{k}\right).$$

Exercise 12

$$k\left(\binom{n}{k}\right) = n\left(\binom{n+1}{k-1}\right).$$

Exercise 13 *For* $k \ge 1$*,*

$$\left(\binom{n}{k}\right) = \sum_{m=1}^{n}\left(\binom{m}{k-1}\right).$$

Although there is no closed form for $\sum_{k=0}^{m}\binom{n}{k}$, $m \ne n$, we have

Exercise 14 *For* $n \ge 0$*,*

$$\sum_{k=0}^{m}\left(\binom{n}{k}\right) = \left(\binom{n+1}{m}\right).$$