Combinatorial Analysis

Pascal's triangle

By using Pascal's formula

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

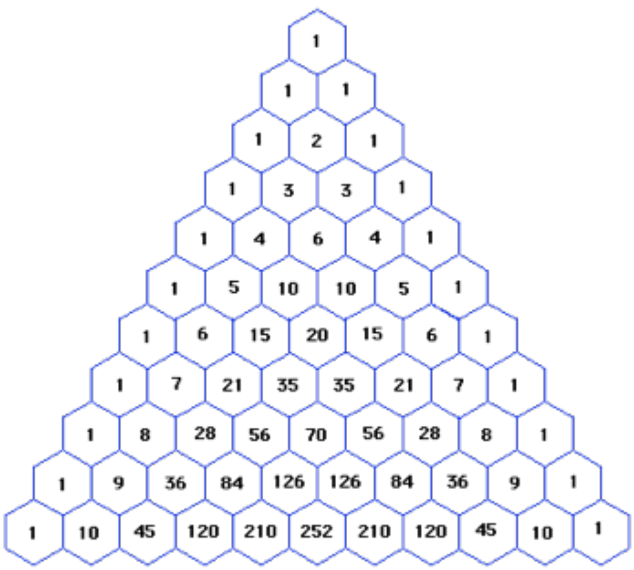
and the initial information

$$\binom{n}{0} = 1, \quad \binom{n}{n} = 1, \quad \text{for } n \geq 0,$$

the binomial coefficients can be calculated recursively without using the expression

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The numbers can be displayed in an infinite array known as Pascal's triangle.



Exercise 1 *Explain how you obtain Pascal's triangle from Pascal's formula.*

Many relations involving binomial coefficients can be discovered by careful examination of Pascal's triangle.

Exercise 2 *Prove that the sum of all numbers in the* $(n+1)$*th row of Pascal's triangle equals* 2^n *for all* $n \geq 0$.

Definition 1 *A triangular number is the number of dots in an equilateral triangle uniformly filled with dots.*

We denote by T_n the nth triangular number.

Exercise 3 *Give a formula for T_n and show that the triangular numbers can be found in Pascal's triangle.*

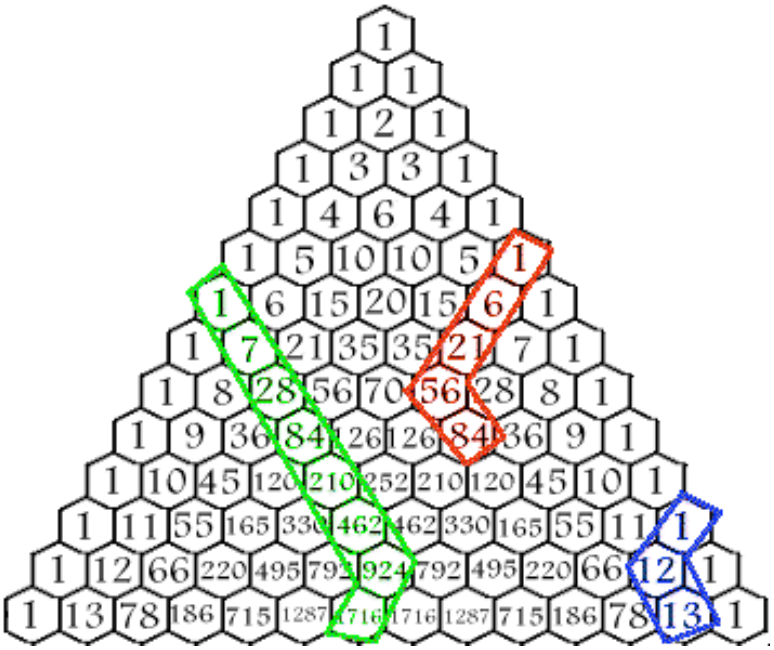
Exercise 4 *Square numbers can also be easily obtained from Pascal's triangle. Prove it.*

Exercise 5 *Fibonacci's sequence can also be located in Pascal's triangle. Prove that, if F_{2m} and F_{2m-1} denote the $(2m)$th and $(2m-1)$th Fibonacci numbers then,*

$$F_{2m} = \binom{2m-1}{0} + \binom{2m-2}{1} + \cdots + \binom{m}{m-1},$$
$$F_{2m-1} = \binom{2m-2}{0} + \binom{2m-3}{1} + \cdots + \binom{m-1}{m-1}.$$

Exercise 6 *Explain why if n is prime, then all the numbers in the $(n+1)$th row (excluding the 1's) are divisible by n. Does this also happen if n is not prime?*

Exercise 7 *(The Hockey Stick Identity) Show that if a diagonal of numbers of any length is selected starting at any of the 1's bordering the sides of the triangle and ending on any number inside the triangle on that diagonal, the sum of the numbers inside the selection is equal to the number below the end of the selection that is not on the same diagonal itself.*



Another interpretation can be given to the entries of Pascal's triangle.

Exercise 8 *Let n be a nonnegative integer and let k be an integer such that $0 \leq k \leq n$. Define $p(n, k)$ as the number of paths from the top of Pascal's triangle $\binom{0}{0}$ to the entry $\binom{n}{k}$, where in each path we move from one entry to the entry in the next row immediately to the right or immediately to the left. Prove that*

$$p(n, k) = \binom{n}{k}.$$

If one examines the binomial coefficients in a row of Pascal's triangle, one notices that the numbers increase for a while and then decrease. A sequence of numbers with this property is called *unimodal*. Thus, the sequence $s_0, s_1, ..., s_n$ is unimodal if there is an integer t with $0 \leq t \leq n$ such that

$$s_0 \leq s_1 \leq \cdots \leq s_t, \quad s_t \geq s_{t+1} \geq \cdots \geq s_n.$$

The number s_t is the largest number in the sequence. The integer t is not necessarily unique because the largest number may occur in the sequence more than once. For instance, if $s_0 = 1$, $s_1 = 3$, $s_2 = 3$, and $s_3 = 2$, then

$$s_0 \leq s_1 \leq s_2, \quad s_2 \geq s_3,$$

but also

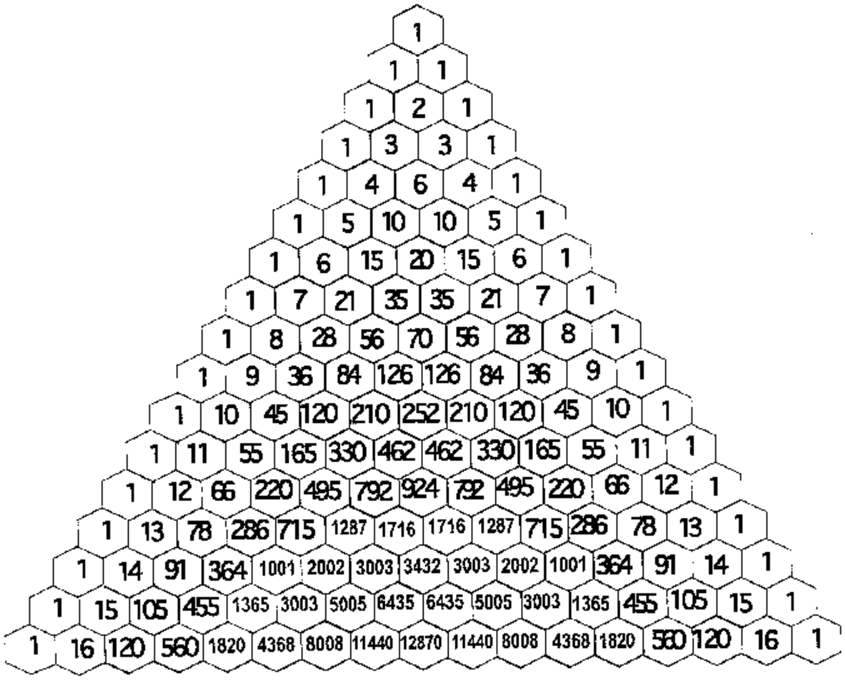
$$s_0 \leq s_1, \quad s_1 \geq s_2 \geq s_3.$$

Exercise 9 *Let n be a positive integer. Show that the sequence of binomial coefficients*

$$\binom{n}{0}, \binom{n}{1}, ..., \binom{n}{n}$$

is unimodal. Also, give the largest binomial coefficient.

Exercise 10 *What is the probability that a randomly selected element of Pascal's triangle is even?*


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 580 120 16 1