Combinatorial Analysis

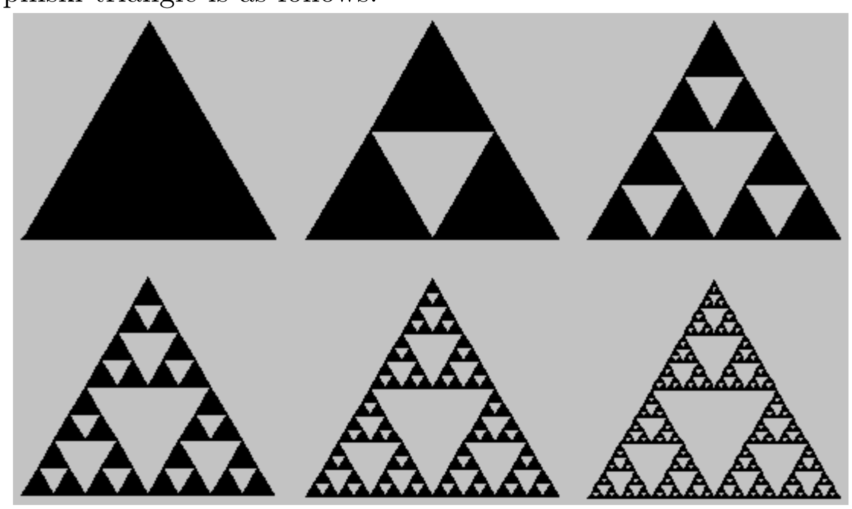
Pascal's triangle

Question: What is the probability that a randomly selected element of Pascal's triangle is even?

Here we are computing the "asymptotic" frequency of even numbers in Pascal's triangle as we increase the number of rows to infinity. We will prove that the ratio of the number of even numbers to the total number of elements approaches 1.

Exercise 1 *How many odd numbers are there in the first 2^n rows? Write a conjecture.*

The Sierpinski triangle, also called Sierpinski gasket or Sierpinski Sieve, is a fractal. This is one of the basic examples of self-similar sets, i.e., mathematically generated patterns that can be reproducible at any magnification or reduction. An algorithm for obtaining arbitrarily close approximations to the Sierpinski triangle is as follows:

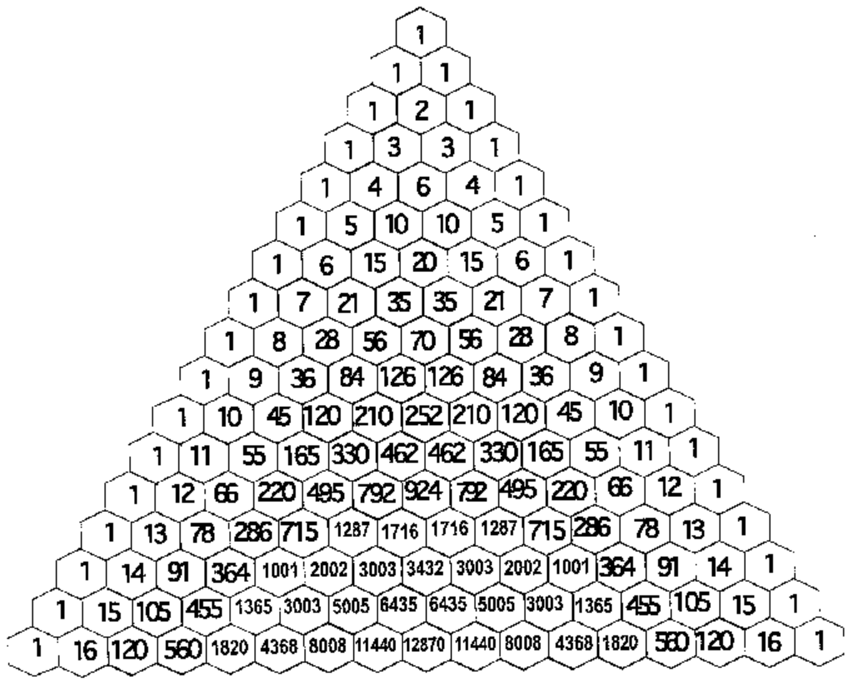


1. Start with any triangle in a plane (any closed, bounded region in the plane will actually work). The canonical Sierpinski triangle uses an equilateral triangle with a base parallel to the horizontal axis (first image).

2. Shrink the triangle to 1/2 height and 1/2 width, make three copies, and position the three shrunken triangles so that each triangle touches the two other triangles at a corner (image 2). Note the emergence of the central hole - because the three shrunken triangles can between them cover only 3/4 of the area of the original. (Holes are an important feature of Sierpinski's triangle.)

3. Repeat step 2 with each of the smaller triangles (image 3 and so on).

Exercise 2 *Consider the first 17 rows of Pascal's triangle below. Shade the odd numbers and compare with the Sierpinski triangle.*



Exercise 3 *Let P_n denote the first 2^n rows of Pascal's triangle. Show that P_{n+1} consists of 3 copies of P_n that surround a triangle of even numbers. In order to do so, use Luca's Theorem below taking $p = 2$.*

Theorem 1 *(Luca's theorem) For non-negative integers m and n and a prime p, the following congruence relation holds:*

$$\binom{m}{n} \equiv \binom{m_0}{n_0}\binom{m_1}{n_1}\cdots\binom{m_k}{n_k} \quad \textit{mod } p,$$

where $m = m_0 + m_1 p + \cdots + m_k p^k$, $n = n_0 + n_1 p + \cdots + n_k p^k$ are the base p expansions of m and n, respectively.

Exercise 4 *(Challenging) Prove Luca's Theorem.*

Exercise 5 *Prove your conjecture in Exercise 1. Use it to compute the probability that a randomly chosen number in Pascal's triangle is even.*

Exercise 6 *Pick a number randomly from the first 2^{2012} rows of Pascal's triangle. To the nearest percent, what is the probability that this number is even?*

Exercise 7 *Use Luca's Theorem to give necessary and sufficient conditions for a binomial coefficient to be divisible by a prime number p. As an application determine if $\binom{38}{17}$ is divisible by 3.*

Exercise 8 *If p divides $\binom{n}{m}$, what is the multiplicity of p in the prime factorization of $\binom{n}{m}$?*