Combinatorial Analysis

Pigeonhole Principle

Here we consider an elementary combinatorial principle that can be used to solve a variety of interesting problems, often with surprising conclusions.

This principle is known under a variety of names, the most common of which are the pigeonhole principle, the Dirichlet box principle, and the shoebox principle.

The pigeonhole principle is usually applied to problems in combinatorial set theory, combinatorial geometry, and number theory.

In its intuitive simplest form, it can be stated as follows: "*If you have more pigeons than pigeonholes, and you try to stuff the pigeons into the holes, then at least one hole must contain at least two pigeons.*".

Dirichlet is credited with first realizing that this simple principle could be used to establish nontrivial results.

Problem 1 *One million trees grow in a forest. It is known that no tree has more than 600,000 leaves. Show that at any moment there are two trees in the forest that have exactly the same number of leaves.*

Theorem 1 *(Pigeonhole principle) Let n and k be positive integers and let $n > k$. Suppose we have to place n identical objects into k identical boxes. Then, there will be at least one box in which we place at least two objects.*

Exercise 1 *Give an argument to show that the pigeonhole principle is true.*

Problem 2 *Thirteen integer numbers are given. Show that there are two numbers among them whose difference is a multiple of 12.*

Problem 3 *Show that in any group of five people, there are two who have the same number of friends within the group.*

Problem 4 *Every point on the plane is colored either red or blue. Prove that no matter how the coloring is done, there must exist two points, exactly one mile apart, that are the same color.*

Problem 5 *Given a unit square, show that if five points are placed anywhere inside or on the border of this square, then two of them must be at most $\sqrt{2}/2$ units apart.*

Problem 6 *There are n married couples. How many of the $2n$ people must be selected in order to guarantee that one has selected a married couple?*

There are other principles related to the pigeonhole principle that are worth to state:

Exercise 2 *If n objects are put into n boxes and no box is empty, how many objects are there in each box?*

Exercise 3 *If n objects are put into n boxes and no box gets more than one object, how many objects are there in each box?*

More abstract formulations of the three principles enunciated thus far are the following. First we recall the following definition.

Exercise 4 *Let $f : X \to Y$ be a function. When is f one-to-one? onto? bijective?*

Problem 7 *Use one of the three principles enunciated before to show that the following result is true: "Let $f : X \to Y$ be a function from X to Y. Assume that X and Y are finite sets. Show that if X has more elements than Y, then f is not one-to-one.*

Problem 8 *Use one of the three principles enunciated before to show that the following result is true: "Let $f : X \to Y$ be a function from X to Y. Assume that X and Y are finite sets. Show that if X and Y have the same number of elements and f is onto, then f is one-to-one.*

Problem 9 *Use one of the three principles enunciated before to show that the following result is true: "Let $f : X \to Y$ be a function from X to Y. Assume that X and Y are finite sets. Show that if X and Y have the same number of elements and f is one-to-one, then f is onto.*

Homework problems (Choose 4 problems among the following problems. Due Thursday.)

Problem 10 *Show that if* $n+1$ *integers are chosen from the set* $\{1, 2, ..., 2n\}$*, then there are always two which differ by 1.*

Problem 11 *In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?*

Problem 12 *Use the pigeonhole principle to show that the decimal expansion of a rational number* m/n *eventually is repeating.*

Problem 13 *Six points are chosen inside a* 3×4 *rectangle. Show that at least two of them are at most* $\sqrt{5}$ *units apart.*

Problem 14 *A chess master who has 11 weeks to prepare a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of (consecutive) days during which the chess master will have played exactly 21 games. (Hint: call* a_i *the number of games that the chess master played the first i days and consider the sequence* $\{a_1, a_2, ..., a_{77}, a_1 + 21, ..., a_{77} + 21\}$*.)*

Problem 15 *(Chinese remainder theorem) Let* m *and* n *be relatively prime positive integers, and let* a *and* b *be integers where* $0 \leq a \leq m-1$ *and* $0 \leq b \leq n-1$*. Then, there is a positive integer* x *such that the remainder when* x *is divided by* m *is* a*, and the remainder when* x *is divided by* n *is* b*, that is*

$$x \equiv a (\text{mod } m), \quad x \equiv b (\text{mod } n).$$

(Hint: Start considering the numbers $\{a, m+a, 2m+a, ..., (n-1)m+a\}$*. Show that they are all different mod* n*.)*