Combinatorial Analysis
Pigeonhole principle

Problem 1 *There are 25 students in a class. Each got an A, B, or a C for a test. Show that there are at least nine students who received the same grade.*

Theorem 1 *(Pigeonhole principle: Generalized form) Let n, m, and r be positive integers such that $n > mr$. If n identical objects are put into m identical boxes, then there will be at least one box into which we place at least r+1 objects.*

Exercise 1 *Give an argument to show that the theorem is true. Using contradiction might be appropriate.*

Problem 2 *Show that among 100 people, there are at least 9 that were born on the same month.*

Theorem 2 *If n objects are placed into m boxes, then at least one box will contain at least $\lceil \frac{n}{m} \rceil$ objects.*

Problem 3 *What is the least number of area-codes needed to guarantee that the 25 million phones in a state have distinct numbers (NXX-NXX-XXXX) where $N \in \{2, 3, ..., 9\}$.*

Problem 4 *Suppose I have 7 blue and 12 black socks in my drawer. It is completely dark and I need to pack for a conference. What is the minimum number of socks I need to take out the drawer to guarantee I have*

(i) at least one matching pair?

(ii) 6 socks of one color?

(iii) 6 black socks?

(iv) 3 matched pairs (possibly including pairs of each color)?

Exercise 2 *Let $f : A \rightarrow B$ be a function where A and B are finite sets. Assume that $|A| > k|B|$. What can you deduce by the application of the Pigeonhole principle in its generalized form?*

Homework Choose 4 problems from the following list.

Problem 5 *Show that if $n + 1$ integers are chosen from the set $\{1, 2, ..., 2n\}$, then there are always two which differ by 1.*

Problem 6 *In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?*

Problem 7 *Use the pigeonhole principle to show that the decimal expansion of a rational number m/n eventually is repeating.*

Problem 8 *Six points are chosen inside a 3×4 rectangle. Show that at least two of them are at most $\sqrt{5}$ units apart.*

Problem 9 *A chess master who has 11 weeks to prepare a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of (consecutive) days during which the chess master will have played exactly 21 games. (Hint: call a_i the number of games that the chess master played the first i days and consider the sequence $\{a_1, a_2, ..., a_{77}, a_1 + 21, ..., a_{77} + 21\}$.)*

Problem 10 *(Chinese remainder theorem) Let m and n be relatively prime positive integers, and let a and b be integers where $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$. Then, there is a positive integer x such that the remainder when x is divided by m is a, and the remainder when x is divided by n is b, that is*

$$x \equiv a (\mathrm{mod}\ m), \quad x \equiv b (\mathrm{mod}\ n).$$

(Hint: Start considering the numbers $\{a, m+a, 2m+a, ..., (n-1)m+a\}$. Show that they are all different mod n.)

The following geometric fact could be helpful in the solution of the next problem: Consider the circumcircle, that is, the circle circumscribed to a triangle (the only circle passing through the three vertices of a triangle). If the lengths of the sides of the triangle are a,b,c, then the radius R of the circumcircle is given by

$$R = \frac{abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}.$$

Problem 11 *Ten points are given within a square of unit size. Prove that we can always find two of them that are closer to each other than 0.48 and we can always find three which can be covered by a disk of radius 1/2.*

Problem 12 *Forty one rooks are placed on a 10×10 chessboard. Prove that there must exist five rooks, none of which attack each other.*

Problem 13 *Two disks, one smaller than the other, are each divided into 200 congruent sectors (slices of a pie). In the larger disk, 100 of the sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk, each sector is painter either red or blue with no stipulation on the number of red and blue sectors. The small disk is then placed on the larger disk so that their centers coincide. Show that it is possible to align the two disks so that the number of sectors of the small disk whose color matches the corresponding sector of the large disk is at least 100.*