Combinatorial Analysis
Pigeonhole principle

Some average principles

- The Pigeonhole principle in Generalized Form is equivalent to: "If the average of m nonnegative integers $n_1, ..., n_m$ is greater than r, that is,

$$\frac{n_1 + \cdots + n_m}{m} > r,$$

then at least one of the integers is greater than or equal to r+1.

- Another averaging principle says: "If the average of m nonnegative integers $n_1, ..., n_m$ is at least equal to r, that is,

$$\frac{n_1 + \cdots + n_m}{m} \geq r,$$

then at least one of the integers is greater than or equal to r.

- And yet one more: "If the average of m nonnegative integers $n_1, ..., n_m$ is less than r+1, that is,

$$\frac{n_1 + \cdots + n_m}{m} < r + 1$$

then at least one of the integers is less than r+1.

Exercise 1 *Show that all average principles can be deduced from the Pigeonhole Principle.*

Theorem 1 *(Pigeonhole principle: Strong form) Let $n, m, q_1, q_2, ..., q_m$ be positive integers. If $n \geq q_1 + q_2 + ... + q_m - m + 1$ identical objects are put into m identical boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects,..., or the mth box contains at least q_m objects.*

Problem 1 *A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?*

Problem 2 *A bag contains 100 apples, 100 bananas, 100 oranges, and 100 pears. If I pick one piece of fruit out of the bag every minute, how long will it be before I am assured of having picked at least a dozen pieces of fruit of the same kind?*

Exercise 2 *Give an argument to show that Theorem 1 is true.*

Exercise 3 *Show that the Pigeonhole principle and the Pigeonhole principle in generalized form are particular cases of the Pigeonhole principle in strong form.*

Problem 3 *Suppose that* $n^2 + 1$ *people are lined up shoulder to shoulder in a straight line. Then, it is always possible to choose* $n + 1$ *of the people to take one step forward so that going from left to right their heights are increasing (or decreasing).*

Problem 4 *Show that every sequence* $a_1, a_2, ..., a_{n^2+1}$ *of* $n^2 + 1$ *real numbers contains either an increasing subsequence of length* $n + 1$ *or a decreasing subsequence of length* $n + 1$*. (Hint: Assume that there is no increasing sequence of length* $n + 1$ *and call* l_k *the length of the longest increasing subsequence beginning with* a_k *for* $k = 1, 2, ..., n^2 + 1$*).*