Combinatorial Analysis
Counting techniques

COUNTING PROBLEMS

A great many counting problems are of the following types:

(i) Count the number of ordered arrangements or ordered selections of objects

  (a) without repeating any object,

  (b) with repetition of objects permitted.

(ii) count the number of unordered arrangements or selections of objects

  (a) without repeating any object,

  (b) with repetition of objects permitted.

Arrangements or selections in (i) in which order is taken into consideration are called *permutations*, whereas arrangements or selections in (ii) in which order is irrelevant are called *combinations*.

For example, my fruit salad is a combination of apples, grapes, and bananas since I do not care in which order the fruits are in. However, a line of people waiting to buy cinema tickets is a permutation.

Exercise 1 *Classify the arrangements or selections in the following examples as permutations with or without repetition or as combinations with or without repetition:*

- *the first three people to win a race,*
- *winning lottery numbers,*
- *combination to the safe,*
- *coins in your pocket.*

Permutations without repetition

Problem 2 *How many ways are there to seat five people in a row of five chairs?*

Problem 3 *How many ways are there to seat five people in a row of eight chairs?*

Problem 4 *How many ways are there to seat eight people in a row of five chairs? In this case, three people will have to remain standing in no particular arrangement.*

Consider a set S with exactly n distinct elements. A *permutation* of S is a linear arrangement (i.e. ordered list) of all n elements of S.

Let r be a positive integer. An *r-permutation* of a set S of n elements is a linear arrangement of exactly r elements of S. We denote by $P(n,r)$ the number of r-permutations of an n-element set.

Exercise 5 *Let $S = \{1,2,3,4\}$. Write all possible 1-permutations, 2-permutations, 3-permutations, 4-permutations and 5-permutations of S.*

Exercise 6 *Let n and r be positive integers. What is $P(n,1)$? Assume that $r > n$. What is $P(n,r)$?*

Problem 7 *Let n and r be positive integers with $r \leq n$. Use the multiplication principle to determine a formula for $P(n,r)$. Prove that your result is true.*

Exercise 8 *What is $P(n,n)$? Give a formula.*

Problem 9 *How many 3-digits numbers can be formed with the integers in $S = \{2,3,4,5,7\}$?*

Extra Problems. Solve the four mandatory problems and three problems of your choice.

Problem 10 *What is the number of different three-letter words we could make using the letters of CHERNOBYL?*

Problem 11 *(Mandatory) What is the number of ways to order the 26 letters of the alphabet so that no two of the vowels a,e,i,o,u occur consecutively?*

Problem 12 *How many orderings are there for a deck of 52 cards if all the cards of the same suit are together?*

Problem 13 *Ten children order ice-cream cones at a store featuring 31 flavors. How many orders are possible in which at least two children get the same flavor?*

The permutations we have considered so far are more properly called *linear permutations*. We think of the objects as being arranged in a line. If instead of arranging the objects in a line, we arrange them in a circle, the number of permutations is smaller.

Problem 14 *Suppose 6 children are marching in a circle. In how many different ways can they form their circle?*

Problem 15 *How many circular permutations of n elements are there? Give an argument to justify your result.*

Problem 16 *How many ways are there to seat five people at a circular table with five chairs? How many ways are there to seat eight people at a circular table with five chairs?*

We call a *circular r-permutation* of a set S of n elements a circular rearrangement of exactly r elements.

Exercise 17 *What are all circular 3-permutations of* $\{1, 2, 3, 4, 5\}$*?*

Problem 18 *(Mandatory) Find a formula for the number of circular r-permutation. Explain.*

Problem 19 *How many ways are there to sit n people at a circular table with r chairs, for $0 \leq r \leq n$?*

Problem 20 *Ten people, including two who do not wish to sit next to each other, are to be seated at a round table. How many circular seating arrangements are there?*

Problem 21 *In how many ways can a necklace be done from 20 beads, each of a different color?*

Problem 22 *(Mandatory) Consider 23 different coloured beads in a necklace. In how many ways can the beads be placed in the necklace so that 3 specific beads always remain together?*

Problem 23 *(Mandatory) In how many ways can 4 married couples seat themselves around a circular table if*

(a) spouses sit opposite each other.

(b) men and women alternate.

Problem 24 *Mom and dad and their 6 children (3 boys and 3 girls) are to be seated at a table. How many ways can this be done if mom and dad sit together and the males and females alternate?*