Lectures on Theory of Numbers

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Chapter 1

Theory of Divisibility

1.1 Divisibility

Definition 1.1.1 (**Natural Number**). *A number used for counting is said to be a natural number. The set of natural numbers is denoted and defined by* $\mathbb{N} = \{1, 2, 3, 4, 5, \cdots\}$.

Definition 1.1.2 (**Integer**). *Any whole number positive, negative, or zero is said to be an integer. The set of integers is denoted and defined by* $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

Definition 1.1.3 (**Divisibility**). *A number* b *is said to be divisible by a number* $a (a \neq 0)$ *if there is a number* c *such that* $b = ac$. *We denote this by* $a \mid b$, *read as* a *divides* b. *If no such integer can be found we write* $a \nmid b$, *read as* a *does not divide* b.

Example 1.1.1. *(1)* $3 \mid 12$ *because* $12 = 3 \cdot 4$

(2) $3 \nmid 7$ *because there is no integer* c *such that* $7 = 3 \cdot c$.

If $a \mid b$, then a is called a divisor of b and b is called a multiple of a. If $a \mid b$ and $0 < a < b$, then a is called a proper divisor or factor of b. We have $b = a \cdot c = (-a) \cdot (-c)$. Thus if $a \mid b$, then $-a \mid b$. For practical purposes we can limit our attention to only positive divisors of integers. 1 is a divisor of any integer and 0 is a multiple of any integer. Any non-zero integer is a divisor or multiple of itself.

Definition 1.1.4 (even number and odd number). *A number which is multiple of 2 is called an even number, otherwise it is called an odd number.*

Theorem 1.1.1. *For any integer* a, b, c *the following hold:*

(i) $a \mid b \Rightarrow a \mid bc$

(ii) $a \mid b$ *and* $b \mid c \Rightarrow a \mid c$

(iii) $a \mid b$ *and* $b \mid a \Rightarrow a = \pm b$

(iv) $a \mid 1 \Rightarrow a = \pm 1$

(v) $a \mid b$ *and* $a \mid c \Rightarrow a \mid (bx + cy)$ *for any integers* x *and* y.

The sum, difference, and product of two integers are obviously integers but the quotient of two integers may or may not be an integer.

Theorem 1.1.2 (**Fundamental Theorem of Divisibility**)**.** *For any integers* $a, b (b \neq 0)$*, there exist unique integers* q *and* r *such that*

$$a = qb + r, \quad 0 \leq r < |b|.$$

Proof. The integer a lies between two consecutive integers of the sequence

$$\cdots, -2|b|, -|b|, 0, |b|, 2|b|, \cdots$$

So WLOG we may assume that $q|b| \leq a < (q+1)|b|$. Then $a - q|b| \geq 0$ and $a - q|b| < |b|$. Let $a - q|b| = r$. Then $0 \leq r < |b|$ implies that

$$\begin{aligned} a &= qb + r \qquad \text{when } b > 0 \\ &= (-q)b + r \qquad \text{when } b < 0 \end{aligned}$$

Hence the existence of q and r is proved.

For uniqueness, let $a = q_1 b + r_1,\ 0 \leq r_1 < |b|$. Then $qb + r = a = q_1 b + r_1, 0 \leq |r_1 - r < |b|$.

$$\begin{aligned} &\Rightarrow (q - q_1)b = r_1 - r \\ &\Rightarrow |q - q_1||b| = |r_1 - r| \\ &\Rightarrow |q - q_1||b| < |b| \\ &\Rightarrow |q - q_1| < 1 \end{aligned}$$

Since q and q_1 are both integers, so the above relation hold if $q - q_1 = 0 \implies q = q_1$.

Consequently, $0 = r_1 - r \Rightarrow r = r_1$.

Thus, q and r are unique.

Hence the theorem. ∎

Definition 1.1.5 (Prime number)**.** *Any integer* p *which exceeds unity is called a prime if it has no integral divisors except* $\pm p$ *and* ± 1*.*

For example 2, 3, 5, etc. are primes.

Definition 1.1.6 (Composite number)**.** *An integer $c > 1$ which has divisors other than $\pm c$ and ± 1 is called a composite number.*

For example 6, 30, 45, etc. are composite numbers.

Note. *1 is neither prime nor composite.*

1.2 Greatest Common Divisor and Least Common Multiple

Definition 1.2.1 (Greatest Common Divisor(g.c.d.))**.** *The greatest integer g that divides both of two integers a and b is called the g.c.d. of a and b and is denoted by $(a, b) = g$.*

For example $(20, 30) = 10$, $(-6, 9) = 3$. Note that $(a, b) \geq 1$, $(a, 0) = |a|$, $(0, 0) = \infty$ and, $(a, a) = a$ for any integer $a \neq 0$.

Definition 1.2.2 (Relatively prime)**.** *If $(a, b) = 1$ then the integers a and b are called relatively prime or co-prime.*

Theorem 1.2.1 (Euclidean Algorithm(330-275 B.C.))**.** *Let a and b be two positive integers where $b \nmid a$. Let $r_0 = a$, $r_1 = b$ and apply the division algorithm repeatedly to obtain a set of remainders $r_2, r_3, \cdots, r_n, r_{n+1}$ defined successively by the relations:*

$$\begin{aligned}
r_0 &= q_1 r_1 + r_2, && 0 < r_2 < r_1 \\
r_1 &= q_2 r_2 + r_3, && 0 < r_3 < r_2 \\
r_2 &= q_2 r_3 + r_4, && 0 < r_4 < r_3 \\
\cdots & \quad \cdots && \cdots \quad \cdots \\
r_{n-2} &= q_{n-1} r_{n-1} + r_n, && 0 < r_n < r_{n-1} \\
r_{n-1} &= q_n r_n + r_{n+1}, && \textit{where} \quad r_{n+1} = 0.
\end{aligned}$$

Then r_n, the last nonzero remainder in this process, is (a, b), the g.c.d. of a and b.

Proof. Since $r_1, r_2, \cdots, r_{n+1}$ are decreasing and nonnegative, so there is a stage at which $r_{n+1} = 0$. The last relation $r_{n-1} = q_n r_n$ shows that $r_n \mid r_{n-1}$ and hence the penultimate relation implies that $r_n \mid r_{n-2}$. By induction we see that $r_n \mid r_i, \forall i$. In particular, $r_n \mid r_1 = b$ and $r_n \mid r_0 = a$, so r_n is a common divisor of a and b.

Now let d be any common divisor of a and b. Then $d \mid a = r_0$ and $d \mid b = r_1$. The definition of r_2 shows that $d \mid r_2$. The next relation shows that $d \mid r_3$. By induction,

$d \mid r_i, \forall i$, so $d \mid r_n$. This shows that $r_n = d$.

Hence r_n is the required g.c.d. of a and b. ∎

The following theorem states that the g.c.d. of any two integers can be expressed as an integral linear combination of them.

Theorem 1.2.2. *If $g = (a, b)$, then there exists integers x and y such that $g = ax + by$.*

Proof. Consider the linear combinations $au + bv$, where $u, v \in \mathbb{Z}$. This set of integers includes positive and negative values, and 0 by the choice of $u = v = 0$. Choose x and y so that $ax + by$ is the least positive integer l in the set. Thus $ax + by = l$.

Now we prove that $l \mid a$ and $l \mid b$. Suppose that $l \nmid a$. Then there exist integers q and r such that

$$a = ql + r,\ 0 < r < l.$$

$$\Rightarrow r = a - ql = a - q(ax + by) = a(1 - qx) + b(-qy).$$

Thus r is in the set $\{au + bv\}$. This contradicts the fact that l is the least positive integer in the set $\{ax + by\}$. Thus $l \mid a$. Similarly, we can show that $l \mid b$.

Now, since g is the g.c.d. of a and b, we may write $a = gA$, $b = gB$, and $l = ax + by = g(Ax + By)$. Thus, $g \mid l$ and so we conclude that $g \leq l$. Now $g < l$ is impossible as g is the g.c.d. of a and b. Thus, $g = l = ax + by$. ∎

Corollary 1.2.3. $(a, b) = 1 \iff \exists\, x, y \in \mathbb{Z}$ *such that* $ax + by = 1$.

Example 1.2.1. *1. Find the g.c.d. of 1769 and 2378 using Euclidean algorithm*

2. Hence express it as an integral linear combination of these numbers.
Solution: 1. We have:

$$\begin{aligned} 2378 &= 1 \cdot 1769 + 609 \\ 1769 &= 2 \cdot 609 + 551 \\ 609 &= 1 \cdot 551 + 58 \\ 551 &= 9 \cdot 58 + 29 \\ 58 &= 2 \cdot 29 \end{aligned}$$

Hence by the Euclidean algorithm $(1769, 2378) = 29$.
2. Now

$$\begin{aligned} 29 &= 551 - 9 \cdot 58 \\ &= 551 - 9 \cdot (609 - 1 \cdot 551) \\ &= 10 \cdot 551 - 9 \cdot 609 \\ &= 10(1769 - 2 \cdot 609) - 9 \cdot 609 \\ &= 10 \cdot 1769 - 29 \cdot 609 \\ &= 10 \cdot 1769 - 29 \cdot (2378 - 1 \cdot 1769) \\ &= 39 \cdot 1769 + (-29) \cdot 2378 \end{aligned}$$

$\therefore 29 = 39 \cdot 1769 + (-29) \cdot 2378$.

Definition 1.2.3 (Least Common Multiple)**.** *The least common multiple(l.c.m.) of two or more non zero integers is the smallest positive integer that is divisible by all of them. If* l *is the l.c.m. of* a *and* b*, we denote it by* $[a, b] = l$.

For example $[20, 30] = 60$. Note that there is no greatest common multiple of two or more integers. Also observe that g.c.d. and l.c.m. are unique.

Theorem 1.2.4. *If* $ab > 0$*, then* $[a, b](a, b) = ab$.

Proof. Let $[a, b] = l$ and $(a, b) = g$. Then $a \mid l$ and $b \mid l$ gives $ab \mid la$ and $ab \mid lb$

$$\therefore ab \mid (la, lb)$$

$$\Rightarrow ab \mid l(a, b)$$

$$\Rightarrow ab \mid lg \quad \cdots (1)$$

On the other hand, since $a \mid \frac{ab}{g}$ and $b \mid \frac{ab}{g}$, so $\frac{ab}{g}$ is a common multiple of a and b. Therefore,

$l \mid \frac{ab}{g} \Rightarrow lg \mid ab \quad \cdots (2)$

From (1) and (2) we conclude that $lg = ab \implies [a, b](a, b) = ab$. ∎

Note. $[a, b]\Big|abm$ *where* m *is an integer i.e.l.c.m. of any two or more integers divides any common multiple of them.*

Theorem 1.2.5. *Let* p *be a prime and* a *be any integer. Then either* $p \mid a$ *or* $(a, p) = 1$.

Proof. If $p \mid a$, then there is nothing to prove. If $p \nmid a$, then we shall prove that $(a, p) = 1$. Let $(a, p) = g$. Then $g \mid a$ and $g \mid p$. But p is a prime, so $g \mid p$ gives $g = 1$ or $g = p$. If $g = p$, then $g \mid a \implies p \mid a$, which is not possible.

Hence $g = 1$ and $(a, p) = 1$. ∎

Theorem 1.2.6. *If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.*

Proof. If $p \mid a$, then there is nothing to prove. If $p \nmid a$, then $(a, p) = 1$. Since $p \mid ab$ and $(a, p) = 1$, so $p \mid b$. ∎

Theorem 1.2.7 (Euclid's Lemma)**.** *If $a \mid bc$ and $(a, b) = 1$, then $a \mid c$.*

Proof. Since $(a, b) = 1$, there exist integers x and y such that

$$ax + by = 1$$

. Since $a \mid bc \Rightarrow bc = am,\ m \in \mathbb{Z}$.

$$\begin{aligned} \therefore c = c \cdot 1 = c(ax + by) &= acx + bcy \\ &= acx + amy \\ &= a(cx + my) \\ &\Rightarrow a \mid c \end{aligned}$$

∎

1.3 Factorization in Prime Numbers

Theorem 1.3.1 (Fundamental Theorem of Arithmetic or Unique Factorization Theorem)**.** *Every positive integer $N > 1$ can be expressed product of primes in one and only one way if we do not distinguish between two same prime factors.*

Proof. Let $p_1 > 1$ be the least divisor of N. Then $p_1 < N$. Evidently, p_1 is a prime. We write $N = p_1 N_1$. If N_1 is a prime, N has been expressed as a product of two primes not necessarily distinct.

If N_1 is composite, its least divisor $p_1 > 1$ is a prime. We write $N_1 = p_2 N_2$, and proceed with N_2 as before. After a finite number of such steps we obtain a factorization

$$N = p_1 p_2 p_3 \cdots p_n$$

of N into primes. Suppose that

$$N = q_1 q_2 q_3 \cdots q_r$$

is a second factorization of N.

Then the prime q_1 evidently divides one of the primes p_i, say p_1. Hence $q_1 = p_1$ and $q_2 q_3 \cdots q_r = p_2 p_3 \cdots p_n$. Similarly q_2 is equal to one of the factors on the right, say p_2. Proceeding in this manner we conclude that $r = n$ and that $q_1, q_2, q_3, \cdots, q_r$ are identical with $p_1, p_2, p_3, \cdots, p_n$ in some order. ∎

Corollary 1.3.2. *An integer $n > 1$ can be uniquely written as $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ where $p_1, p_2, p_3, \cdots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_k$ are positive integers.*

Proof. By 1.3.1 we have

$$n = p_1 p_2 p_3 \cdots p_r$$

Since the prime factors are not necessarily distinct, by collecting the same primes we can write the above equation as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}.$$

∎

This representation is called the **canonical representation** or the **standard representation** or the **standard factorization** of n.

Motivation:. $N = 2 \cdot 3 \cdot 5 = 30$
$P = N + 1 = 30 + 1 = 31$, *not divisible by the prime in the composition of P.*

Theorem 1.3.3 (Euclid)**.** *The number of primes is infinite.*

Proof. Let there are only a finite number of primes, say $p_1, p_2, p_3, \cdots, p_k$, where p_k is the largest prime. We define

$$N = p_1 p_2 p_3 \cdots p_k$$

$$\Rightarrow N + 1 = p_1 p_2 p_3 \cdots p_k + 1$$

$$\Rightarrow P = p_1 p_2 p_3 \cdots p_k + 1, \text{where } P = N + 1.$$

Now if P is a prime then p_k is not the largest prime. If P is not a prime, then by Fundamental Theorem of Arithmetic, P can be factored into prime factors. So there exist a prime p such that $p \mid P$.

Since there are only prime $p_1, p_2, p_3, \cdots, p_k$, so p will be one of them, say $p = p_i,\ 1 \leq i \leq k$. Now $p \mid P \Rightarrow p_i \mid P$, but this impossible by definition of P. Hence it is a contradiction. Thus the number of primes is infinite. ∎

Example 1.3.1. *Show that one of the two consecutive integers is divisible by 2.*

Solution: Let the two consecutive integers be $n, (n+1)$. If n be odd, then $n+1$ is even and is divisible by 2. If n is even then it is divisible by 2.

Example 1.3.2. *Show that one of the three consecutive integers is divisible by 3.*

Solution: Let $n, n+1, n+2$ be any three consecutive integers. Here n must be any one of the forms $3m, 3m+1, 3m+2$. When $n = 3m$, the first integer is divisible by 3, when $n = 3m+1,\ n+2 = 3m+3$ which is divisible by 3, when $n = 3m+2,\ n+1 = 3m+3$ which is divisible by 3.

Example 1.3.3. *Show that the product of any three consecutive integers is divisible by 6.*

Solution: Let n be any integers. Then $n(n+1)(n+2)$ is divisible by 6.

$a^5 - 1 = (a-1).(a^4 + a^3 + 1)$

Example 1.3.4. *If $2^p - 1$ is a prime, then show that p is itself a prime.*

Solution: Suppose that p is not a prime. Then it is composite, say $p = ab$. Then $2^p - 1 = 2^{ab} - 1 = (2^a)^b - 1$ which is divisible by $2^a - 1$. So, $2^{ab} - 1$ or $2^p - 1$ is a composite number which contradicts our given assumption.

So p must be a prime.

Example 1.3.5. *If $2^n + 1$ is a prime, then n is a power of 2.*

Solution: If n is not a power of 2, then suppose that $n = 2^m \times q$ where q is odd and greater than 1. Now $2^{2^m \times q} + 1 = (2^{2^m})^q + 1$ which is divisible by $2^{2^m} + 1$. This contradicts that $2^n + 1$ is a prime. Hence n must be a power of 2.

1.4 The functions $\tau, \sigma, \mathbf{P}$, and σ_k

Definition 1.4.1. *The function*

$$\tau : \mathbb{N} \to \mathbb{N} \text{ defined by } \tau(n) = \sum_{\substack{d \mid n \\ d \geq 1}} 1$$

denotes the number of all positive divisors of n, where n is a positive integer.

Note. *Some authors write d(n) for $\tau(n)$.*

Definition 1.4.2. *The function*

$$\sigma : \mathbb{N} \to \mathbb{N} \text{ defined by } \sigma(n) = \sum_{\substack{d|n \\ d\geq 1}} d$$

denotes the sum of all positive divisors of n, where n is a positive integer.

Definition 1.4.3. *The function*

$$\mathbf{P} : \mathbb{N} \to \mathbb{N} \text{ defined by } \mathbf{P}(n) = \prod_{\substack{d|n \\ d\geq 1}} d$$

denotes the product of all positive divisors of n, where n is a positive integer.

Definition 1.4.4. *The function*

$$\sigma_k : \mathbb{N} \to \mathbb{N} \text{ defined by } \sigma_k(n) = \sum_{d|n} d^k$$

denotes the sum of the k-th powers of the divisors of n, where n is a positive integer.

Example 1.4.1. *Following table lists $\tau(n), \sigma(n)$ and $\mathbf{P}(n)$ for $n = 1, 2, 3, \cdots, 10$.*

n	1	2	3	4	5	6	7	8	9	10
$\tau(n)$	1	2	2	3	2	4	2	4	3	4
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18
$\mathbf{P}(n)$	1	2	3	8	5	36	7	64	27	100

Table 1.1: $\tau(n), \sigma(n)$ and $\mathbf{P}(n)$

Example 1.4.2. *Let n=6. The divisors of 6 are $d : 1, 2, 3, 6$.*
$\sigma_1(n) = \sigma(n) = \sum_{d|n=6} d = 1 + 2 + 3 + 6 = 12$
$\sigma_0(n) = 1^0 + 2^0 + 3^0 + 6^0 = 1 + 1 + 1 + 1 = 4 = \tau(n)$
$\sigma_3(n) = \sum_{d|n=6} d^3 = 1^3 + 2^3 + 3^3 + 6^3 = 1 + 8 + 27 + 216 = 252$

Remark. *If n is a prime, then n has only two positive divisors, 1 and n itself.*

$$\therefore \tau(n) = 2, \sigma(n) = n + 1, \text{ and } \mathbf{P}(n) = n.1 = n$$

If n is composite then the general formulae for τ, σ and $\mathbf{P}$ are given by the following theorems:

Theorem 1.4.1. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ where $p_1, p_2, p_3, \cdots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_k$ are positive integers, then*

(i)

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) = \prod_{i=1}^{k} (\alpha_i + 1)$$

(ii)

$$\sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1}\cdots\frac{p_k^{\alpha_k+1}-1}{p_k-1} = \prod_{i=1}^{k}\frac{p_i^{\alpha_i+1}-1}{p_i-1}$$

Proof. We shall prove the theorem by induction on k. Let $k=1$, that is, n has only one prime factor, say $n=p^\alpha$, where α is a positive integer. Then the positive divisors of n are $1, p, p^2, \cdots, p^\alpha$. We have

$$\tau(n) = \alpha+1 \text{ and } \sigma(n) = 1+p+p^2+\cdots+p^\alpha = \frac{p^{\alpha+1}-1}{p-1}$$

Next suppose that (i) and (ii) hold when n has $k-1$ distinct prime factors($k \geq 2$). Write $n = n'p_k^{\alpha_k}$ where $n' = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_{k-1}^{\alpha_{k-1}}$, $(n', p_k^{\alpha_k}) = 1$, and all p's are distinct primes. Moreover, any divisor of n' is a divisor of n. Hence any divisor of n is of the form $d'p_k^t,\ 0 \leq t \leq \alpha_k,\ d' \geq 1,\ d' \mid n'$.

$$\begin{aligned}
\tau(n) &= \sum_{\substack{d|n\\ d\geq 1}} 1 \\
&= \sum_{d'|n} 1 + \sum_{d'p_k|n} 1 + \sum_{d'p_k^2|n} 1 + \cdots + \sum_{d'p_k^{\alpha_k}\big| n} 1 \\
&= \underbrace{\tau(n') + \tau(n') + \tau(n') + \cdot + \tau(n')}_{\alpha_k+1 \text{ terms}} \\
&= \tau(n')(\alpha_k+1)
\end{aligned}$$

By induction hypothesis we have $\tau(n') = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_{k-1}+1)$

$\therefore \tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_{k-1}+1)(\alpha_k+1) = \prod_{i=1}^{k}(\alpha_i+1)$

$$\begin{aligned}
\sigma(n) &= \sum_{\substack{d|n\\ d\geq 1}} d \\
&= \sum_{\substack{d'|n'\\ d'\geq 1}} d' + \sum_{\substack{d'|n'\\ d'\geq 1}} d'p_k + \cdots + \sum_{\substack{d'|n'\\ d'\geq 1}} d'p_k^{\alpha_k} \\
&= \left(\sum_{\substack{d'|n'\\ d'\geq 1}} d'\right) + \left(\sum_{\substack{d'|n'\\ d'\geq 1}} d'\right) p_k + \cdots + \left(\sum_{\substack{d'|n'\\ d'\geq 1}} d'\right) p_k^{\alpha_k} \\
&= \left(\sum_{\substack{d'|n'\\ d'\geq 1}} d'\right)(1+p_k+\cdots+p_k^{\alpha_k}) \\
&= \sigma(n')\frac{p_k^{\alpha_k+1}-1}{p_k-1}
\end{aligned}$$

By induction hypothesis

$$\sigma(n') = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdots \frac{p_{k-1}^{\alpha_{k-1}+1}-1}{p_{k-1}-1}$$

$$\therefore \sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdots \frac{p_{k-1}^{\alpha_{k-1}+1}-1}{p_{k-1}-1} \cdot \frac{p_k^{\alpha_k+1}-1}{p_k-1} = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i-1}$$

■

1. If $a = qb + r$ then prove that $(a, b) = (b, r)$.

2.

Chapter 2

Congruences

2.1 Congruences and Its Properties

The concept of congruences was introduced by Carl Friedrich Gauss about 1800, one of the greatest mathematician of all time. Congruence often arises in every day life. For instance, if the second of January is Sunday, then 9,16,23 of the same month are all Sundays, since when they are divided by 7, the remainders are all 2.

A congruence is nothing but a refined statement about divisibility. Congruence can be treated in the same manner as equations. Let m be a positive integer. Two integers a and b leave remainders when divided by m. If the remainders are same we say that a is congruent to be modulo m, and we write

$$a \equiv b(\text{mod} m) \qquad \text{or} \qquad m \mid a - b.$$

Alternatively,

Definition 2.1.1 (Congruence)**.** *If the difference of two integers a and b is divisible by m, we shall say that a is congruent to b modulo m, and we write*

$$a \equiv b(mod\ m) \qquad or \qquad m \mid a - b.$$

Definition 2.1.2 (Incongruence)**.** *If the difference of two integers a and b is not divisible by m, we shall say that a is incongruent to b modulo m, and we write*

$$a \not\equiv b(mod\ m) \qquad or \qquad m \nmid a - b.$$

Example 2.1.1. $7 \equiv 3(mod\ 2)$, $9 \not\equiv 4(mod\ 3)$, *and* $13 \equiv 3(mod\ 5)$.

Theorem 2.1.1. *If $a \equiv b(mod\ m)$ and $c \equiv d(mod\ m)$, then* *(i)* $a + c \equiv b + d(mod\ m)$

(ii) $a - c \equiv b - d(mod\ m)$

(iii) $ac \equiv bd(mod\ m)$

Proof. Here $a - b$ and $c - d$ are both multiples of m, say $a - b = t_1 m$ and $c - d = t_2 m$.

(i) $(a-b)+(c-d) = (a+c)-(b+d) = (t_1+t_2)m = t_3 m$, say. So, $a+c \equiv b+d(\text{mod } m)$.

(ii) $(a-c)-(b-d) = (a-b)-(c-d) = (t_1-t_2)m = t_4 m$, say. So, $a-c \equiv b-d(\text{mod } m)$.

(iii) $a - b = t_1 m \Rightarrow (a-b)c = t_5 m$ where $t_5 = t_1 c$. So, $ac - bc = t_5 m \cdots (A)$

$c - d = t_2 m \Rightarrow (c-d)b = t_6 m$ where $t_6 = t_2 b$. So, $bc - bd = t_6 m \cdots (B)$ Then from (A) and (B), $ac - bc + bc - bd = (t_5 + t_6)m \Rightarrow ac - bd = t_7 m$, say. So, $ac \equiv bd(\text{mod } m)$.

∎

Remark. *$a \equiv b(mod\ m) \Rightarrow na \equiv nb(mod\ m)$, but the converse need not hold.*

e.g., $4 \cdot 7 \equiv 4 \cdot 2(\text{mod } 10) \Rightarrow 7 \not\equiv 2(\text{mod } 10)$ but $7 \equiv 2(\text{mod } 5)$.

Theorem 2.1.2 (Cancelation law)**.** *If $na \equiv nb(mod\ m)$ and $(n, m) = g$, then $a \equiv b\left(mod\ \frac{m}{g}\right)$*

Proof. Since $(n, m) = g$, we have $n = gN$ and $m = gM$ where $(M, N) = 1$. Given that $m \mid n(a - b) \Rightarrow gM \mid gN(a - b) \Rightarrow M \mid N(a - b) \Rightarrow M \mid a - b$ Therefore, $a \equiv b(\text{mod } M)$ and hence $a \equiv b\left(\text{mod } \frac{m}{g}\right)$. ∎

Corollary 2.1.3. *If a, b, g are integers such that $na \equiv nb(mod\ m)$ and $(m, n) = 1$, then $a \equiv b(mod\ m)$.*

Definition 2.1.3. *Let $a, b(b > 0)$ be any two integers. Then $\exists$ integers q, r such that $a = bq + r$ where $0 \leq r < b$. Then r is called the least residue of a modulo m.*

Here $r := 0, 1, 2, \cdots, b - 1$ form a complete set of least residues modulo b.

Motivation:. *Let $r = 6$ be any integer, $a = 3$, $b = 5$, and $(a, b) = 1$. Then the least residues modulo b of*

r,	a+r,	2a+r,	3a+r,	4a+r	are
1,	4,	2,	0,	3	respectively.

Theorem 2.1.4 (Least Residue Theorem)**.** *If a and $b(b > 0)$ are relatively prime and if r is any integer, then the least residues modulo b of*

$$r, a+r, 2a+r, 3a+r, \cdots, (b-1)a+r \cdots (1)$$

are

$$0, 1, 2, 3, \cdots, (b-1) \cdots (2)$$

in some rearranged order.

Proof. By definition, the least residue modulo b of any integer is one of the numbers (2). There are b numbers (1) and b numbers (2). The theorem will be established if we can show that no two of the numbers (1) have the same least residue (2). Any two of the numbers of (1) may be denoted by

$$sa + r,\ 0 \leq s < b \text{ and } ta + r,\ 0 \leq t < b \text{ with } s > t.$$

If they have the same least residue modulo b, they are congruent. Then

$$sa + r \equiv ta + r (\text{mod } b)$$

$$sa \equiv ta (\text{mod } b)$$

$$s \equiv t (\text{mod } b), \text{ since } (a, b) = 1$$

But $s - t$ is positive and less than b so that it is not divisible by b. This contradiction shows that no two distinct numbers (1) have the same least residue modulo b. Hence the theorem is proved. ∎

Corollary 2.1.5. *If $(a, b) = 1$, then $a, 2a, 3a, \cdots, (b-1)a$ are congruent modulo b to $1, 2, 3, \cdots, (b-1)$ in some order.*

Proof. Since $(a, b) = 1$, no one of ia is divisible by b; $i := 1, 2, 3, \cdots, b-1$. So, no residue is zero. Thus, the corollary is established. ∎

Theorem 2.1.6 (Farmat's Theorem)**.** *If p is a prime and $(a, p) = 1$, then $a^{p-1} \equiv 1(mod\ m)$.*

Proof. By the least residue theorem the integers $a, 2a, 3a, \cdots, (p-1)a$ are congruent modulo p to $1, 2, 3, \cdots, p-1$ in some order. The product of the first set is congruent to that for the second set. Then

$$a^{p-1} 1 \cdot 2 \cdot 3 \cdot \cdots \cdot p-1 \equiv 1 \cdot 2 \cdot 3 \cdot \cdots \cdot p-1 (\text{mod } p)$$

$$a^{p-1}(p-1)! \equiv (p-1)! (\text{mod } p)$$

$$a^{p-1} \equiv 1 (\text{mod } p), \text{ since } ((p-1)!, p) = 1$$

Hence the theorem is proved. ∎

Definition 2.1.4. *Let $n \geq 1$ be a positive integer. Then $\phi(n)$ denotes the number of numbers not exceeding n, which are relatively prime to n. This $\phi(n)$ is called Euler's ϕ function.*

Example 2.1.2. $\phi(1) = 1, \phi(2) = 1, \phi(4) = 2, \phi(5) = 4, \phi(6) = 2, \phi(7) = 6, \phi(8) = 4, \phi(9) = 6, \phi(10) = 4, \phi(11) = 10, \phi(12) = 4, \phi(13) = 12, \phi(14) = 6$.

Motivation:. *The positive integers not exceeding $m = 8$ which are relatively prime to $m = 8$ are $a := 1, 3, 5,$ and 7.*
Now construct the following congruences with $a = 3$: $3.1 \equiv 3, \quad 3.3 \equiv 1, \quad 3.5 \equiv 7, \quad 3.7 \equiv 5 (mod\ m)$
Multiplying $3^4.1.3.5.7 \equiv 3.1.7.5 (mod\ m) \Rightarrow 3^4 \equiv 1 (mod\ m),$ since $(1.3.5.7, m) = 1$
$\therefore 3^{\phi(8)} \equiv 1 (mod\ m) \Rightarrow a^{\phi(m)} \equiv 1 (mod\ m)$ *where $(a, m) = 1$. This result is known as the Euler's theorem.*

Theorem 2.1.7.

Proof.

∎

1. Prove that congruence relation is an equivalence relation.

2.

Chapter 3

Sets, The Real Number System, and Functions

3.1 Sets

Definition 3.1.1 (**Set**). *A set is a well-defined list, collection, or class of objects.*

Examples include the following:

Example 3.1.1. *(a)* $A = \{1, 5, 8, 10\}$

(b) $B = \{2n : n \in N\}$

Does EVERY collection of objects make up a set?

The set operations are binary operations and are some useful tools to obtain new sets from the old ones through combining(union), intersecting(intersection), and taking differences(difference or complement).

Definition 3.1.2 (**Union of two sets**). *The union of two sets A and B is mathematically defined as*

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or } x \in \text{both}\}.$$

Definition 3.1.3 (**Intersection of two sets**). *The intersection of two sets A and B is mathematically defined as*

$$A \cap B = \{x : x \in A, x \in B\}.$$

Definition 3.1.4 (**Difference of two sets**). *The difference of two sets A and B is mathematically defined as*

$$A \sim B = \{x \in A : x \notin B\}.$$

The difference of A and B is often termed as the complement of B relative to A

Definition 3.1.5 (**Venn diagram**). *Venn diagram is a convenient way to depict the relations among the sets.*

3.1.1 Cartesian Product Sets and their visualization

Definition 3.1.6 (**Cartesian Product set**). *The cartesian product of two nonvoid sets A and B is mathematically defined as*

$$A \times B = \{(a, b) : a \in A,\ b \in B\}.$$

Problem 3.1.1. *Visualize $A \times B$ where $A = \{x : 1 \leq x \leq 2\}$ and $B = \{x : 0 \leq x \leq 1 \text{ or } 2 \leq x \leq 3\}$.*

3.2 The Real Number System

3.2.1 The Field Properties and the Order Properties of $\mathbb{R}$

The real number system $\mathbb{R}$ can be described as a complete ordered field.

- The field properties of $\mathbb{R}$, often called the algebraic properties, are based on the two operations of addition and multiplication.
- The order properties of $\mathbb{R}$ are based on inequalities of numbers.
- The completeness properties of $\mathbb{R}$ are based on some completeness axioms which will be discussed later.

The historical development of real number system $\mathbb{R}$ was from the positive integers, called the natural numbers $\mathbb{N}$ to the rational numbers $\mathbb{Q}$ to the full real numbers $\mathbb{R} := \mathbb{Q} \cup \mathbb{Q}'$, where the irrational numbers $\mathbb{Q}'$ were obtained either as cuts of rationals (Dedekind), or as Cauchy sequences or as suprema of sets of rationals (Supremum property of $\mathbb{R}$). The existence of irrationals is guaranteed by any of the three theorems called the **Completeness Axiom of** $\mathbb{R}$.

Definition 3.2.1 (**Field Properties of $\mathbb{R}$**). *On the set of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:*

(A1) closure property of addition

(A2) commutative property of addition

(A3) associative property of addition

(A4) existence of an additive identity, called zero element, in $\mathbb{R}$

(A5) existence of an additive inverse, called negative element, in $\mathbb{R}$

(M1) closure property of multiplication

(M2) commutative property of multiplication

(M3) associative property of multiplication

(M4) existence of a multiplicative identity, called unit element, in $\mathbb{R}$

(M5) existence of a multiplicative inverse, called reciprocals, in $\mathbb{R}$

(D) distributive property of multiplication over addition

The real number system $\mathbb{R}$ is a **FIELD** because it obeys the field axioms.

Definition 3.2.2 (**Order Properties of** $\mathbb{R}$)**.** *On the field of real numbers there is a binary relation denoted by* $<$*. For all* x, y, $z \in \mathbb{R}$ *this relation satisfies the following properties:*

(O1) Law of Trichotomy: either $x < y$ *or* $x = y$ *or* $y < x$

(O2) Transitive Law: $x < y$ *and* $y < z$ *implies* $x < z$

(O3) compatibility of $<$ *and* $+$ *:* $x < y$ *implies* $x + z < y + z$

(O4) compatibility of $<$ *and* $\cdot$ *:* $0 < z$ *and* $x < y$ *implies* $x.z < y.z$

The real number system $\mathbb{R}$ is an **ORDERED FIELD** because it obeys the order axioms.

3.2.2 The Completeness Properties of $\mathbb{R}$

Theorem 3.2.1. $\sqrt{2}$ *is not a rational number.*

Proof. If $\sqrt{2}$ be a rational number, assume $\sqrt{2} = \frac{m}{n}$ where $(m, n) = 1$ and $m, n \in \mathbb{Z}$, $n \neq 0 \implies m^2 = 2n^2 \implies m$ is an even integer, so $m = 2k$, say. So $(2k)^2 = 2n^2 \implies n^2 = 2k^2 \implies n$ is an even integer, a contradiction to the assumption. Hence $\sqrt{2}$ is not a rational number. ∎

Definition 3.2.3 (**Intervals**)**.** *Open interval:* $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ *and closed interval:* $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Definition 3.2.4 (**Bounded Set**). *Let $S \subset \mathbb{R}$. Then $u \in \mathbb{R}$ is said to be an upper bound of S if $s \leq u$ for all $s \in S$ and $l \in \mathbb{R}$ is said to be a lower bound of S if $l \leq s$ for all $s \in S$. Set S is bounded above if it has an upper bound and bounded below if it has a lower bound. It is said to be bounded if it has both upper bound and lower bound. It is unbounded if it lacks either an upper bound or a lower bound.*

Definition 3.2.5 (**Suprema and Infima**). *If S is bounded above, then an upper bound $u \in \mathbb{R}$ is said to be a least upper bound (lub) or supremum of S if no number smaller than u is an upper bound of S. Similarly greatest lower bound(glb) or infimum is defined.*

Theorem 3.2.2. *An upper bound u of a nonempty set $S \subseteq \mathbb{R}$ is the supremum of S if and only if for each $\epsilon > 0$ there exists an $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.*

Example 3.2.1.

- *If $S = (0, 1)$, the $\sup S = 1$, and $\inf S = 0$.*
- *If $S = [0, 2]$ the $\sup S = 2$, greatest element of S and $\inf S = 0$, the least element os S.*
- *If $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, the $\sup S = 1$, greatest element of S and $\inf S = 0$. But the set has no least element.*

Example 3.2.2. *Since every real number is an upper bound for the empty set, so the empty set has no supremum. Similarly it has no infimum.*

Theorem 3.2.3 (**Completeness Axiom of $\mathbb{R}$/The Supremum Property of $\mathbb{R}$**). *Every nonempty set of real numbers that has an upper bound has a supremum in $\mathbb{R}$.*

Theorem 3.2.4 (**Completeness Axiom of $\mathbb{R}$/The Infimum Property of $\mathbb{R}$**). *Every nonempty set of real numbers that has a lower bound has an infimum in $\mathbb{R}$.*

Theorem 3.2.5. *There exists a positive real number x such that $x^2 = 2$.*

Proof. Let $S = \{s \in \mathbb{R} : 0 \leq s,\ s^2 < 2\}$. Since $1 \in S$, the set is not empty. Also, S is bounded above by 2, because if $t > 2$, then $t^2 > 4$ so that $t \notin S$. Therefore the Supremum Property implies that the set S has a supremum in $\mathbb{R}$ and let $x = \sup S$(here $\sup S = \sqrt{2}$).

Thus after some mathematical reasoning we can prove that x is a real number. ∎

Definition 3.2.6. *A* **cut** *or* **Dedekind cut** *of the real line is a pair of nonempty subsets L and R of $\mathbb{R}$ such that $L \cup R = \mathbb{R}$ and for every $x \in L$ and $y \in R$, $x < y$. In terms of geometry, L is the left-hand set and R is the right-hand set.*

Example 3.2.3. *If $(-\infty, 1] \cup (1, \infty) = \mathbb{R}$, then $L = (-\infty, 1]$ and $R = (1, \infty)$*

Theorem 3.2.6 (**Completeness Axiom of $\mathbb{R}$**). *Let L and R define the cut of the real line. Then there is one and only one real number z such that for every $x \in L$ and $y \in R$, $x \leq z \leq y$.*

Example 3.2.4. *We have earlier proved that $\sqrt{2}$ is not a rational number and observe here that $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty) \neq \mathbb{R}$, but $(-\infty, \sqrt{2}) \cup \{\sqrt{2}\} \cup (\sqrt{2}, \infty) = \mathbb{R}$. So by the Dedekind theorem $\sqrt{2}$ is a real number.*

Thus, from the above discussion we may conclude that $\mathbb{R}$ is complete **BUT** $\mathbb{Q}$ is not.

3.3 Functions

Definition 3.3.1 (**Function**)**.** *A function f from a set A into a set B, denoted by $f : A \to B$, is a rule that assigns to each element $x \in A$ a unique element $y \in B$, and we write $y = f(x)$.*

Or more precisely, A function f from a set A into a set B, denoted by $f : A \to B$, is a subset f of the cartesian product $A \times B$ such that for each $x \in A$ there exists a unique $y \in B$, and we write $y = f(x)$. Set A is called the **domain** of f and the set $\{y \in B : y = f(x)\}$ is called the **range** of f.

Example 3.3.1. *The equation $y = 2x + 1$ defines a function but $x^2 + y^2 = 1$ does not.*

Definition 3.3.2 (**Monotone function**)**.** *A function f is said to be increasing if $f(x_1) \leq f(x_2)$ for $x_1 \leq x_2$ and decreasing if $f(x_1) \geq f(x_2)$ for $x_1 \leq x_2$. A function is said to be monotone if it is either increasing or decreasing.*

Definition 3.3.3 (**Injective/one-one function**)**.** *A function $f : A \to B$ is said to be one-one if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.*

Definition 3.3.4 (**Surjective/onto function**)**.** *A function f is said to be onto if $f(A) = B$.*

Definition 3.3.5 (**Bijective function/1-1 correspondence**)**.** *A function f is said to be bijective if it is both injective and surjective.*

Definition 3.3.6 (**Inverse function**)**.** *A function, denoted by f^{-1}, is said to be the inverse of f if f is one-one. Thus f^{-1} is related to f as follows:*

$$x = f^{-1}(y) \qquad \text{if and only if} \qquad y = f(x).$$

PROBLEM PLUS 3

1.

2.

Chapter 4

Sequence and Sequence of Functions

4.1 Sequence

Definition 4.1.1 (**Sequence**). *A sequence in $\mathbb{R}$ is a function from the set $\mathbb{N}$ of natural numbers into the set $\mathbb{R}$ of real numbers. Thus, if $X : \mathbb{N} \to \mathbb{R}$ is a sequence, then the value of X at $n \in \mathbb{N}$ is denoted by x_n rather than $X(n)$. We denote sequence by the notations:*

$$X, \qquad (x_n), \qquad < x_n > .$$

What is the difference between the **SET** and **SEQUENCE**?

Definition 4.1.2 (**Convergence of Sequence**). *A real number x is said to be a limit of a sequence x_n, written as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ if for every $\epsilon > 0$ there exists a natural number $K := K(\epsilon)$ such that $|x_n - x| < \epsilon$ for all $n \geq K$. If x is the limit of a sequence, we say that x_n converges to x or x_n has the limit x. If a sequence has a limit we say that the sequence is convergent; if it has no limit, we say that the sequence is divergent.*

Example 4.1.1. *(i) $x_n = \frac{1}{n}$ is a convergent sequence.*

(ii) $x_n = \frac{2n}{n+2}$ is a convergent sequence.

(ii) $X =< 2, 4, 6, 8, ... >$ is a divergent sequence.

(iv) $x_n = (-1)^n$ is a bounded but divergent sequence.

Theorem 4.1.1. *Every convergent sequence is bounded.*

Example 4.1.2. $x_n = \frac{2n}{n+2}$

Theorem 4.1.2 (**Archimedean Property**). *If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x < n_x$.*

This suggests that $\mathbb{N}$, the set of natural numbers is unbounded.

Example 4.1.3. *Show that* $\lim(1/n) = 0$.

Solution: For given $\epsilon > 0$, $1/\epsilon > 0$. *Hence by Archimedean Property there exists a natural number* $K := K(\epsilon)$ *such that* $1/\epsilon < K$, *then for any* $n \in \mathbb{N}$ *such that* $n \geq K \implies 1/\epsilon < K \leq n$ *so that* $1/n < \epsilon$. *Therefore, if* $n \geq K$, *then* $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$.

Theorem 4.1.3. *The sum, difference, scalar multiple, product, and division (not always) of two convergent sequences are convergent.*

Theorem 4.1.4 (**Squeeze Theorem**)**.** *If* $x_n \leq y_n \leq z_n$ *for* $n \geq K$ *and both* x_n, $z_n \to x$, *then* $y_n \to x$

Example 4.1.4. *Use the above theorem to show that* $\lim(\frac{\sin n}{n}) = 0$

Theorem 4.1.5 (**Absolute Convergence**)**.** *If* $|x_n| \to 0$, *then* $x_n \to 0$.

Example 4.1.5. *Use the above theorem to show that* $\lim(\frac{(-1)^n}{n}) = 0$

Theorem 4.1.6 (**Ratio Test**)**.** *Let* x_n *be a sequence of positive real numbers such that* $L = \lim(x_{n+1}/x_n)$ *exists. If* $L < 1$, *then* x_n *converges and* $\lim(x_n) = 0$.

Example 4.1.6. *Use the above theorem to show that* $\lim(n!/n^n) = 0$

Theorem 4.1.7 (**Monotone Convergence Theorem**)**.** *A monotone sequence of real numbers is convergent iff it is bounded. Further, if* $< x_n >$ *is a bounded increasing sequence, then* $x_n \to \sup\{x_n\}$ *and if* $< x_n >$ *is a bounded decreasing sequence, then* $x_n \to \inf\{x_n\}$

Example 4.1.7. *Consider the sequence* $x_{n+1} = (x_n + a/x_n)/2$ *where* $a > 0$, $x_1 > 0$. *(This calculates the square roots of* a*). Here* $x_n - x_{n+1} \geq 0 \implies x_{n+1} \leq x_n$ *for all* $n \geq 2$. *It follows from Monotone Convergence Theorem that* $x = \lim(x_n)$ *exists. So* $x = (x + a/x)/2 \implies x = \sqrt{a}$.

Theorem 4.1.8 (**The Bolzano-Weierstrass Theorem**)**.** *A bounded sequence of real numbers has a convergent subsequence*

Example 4.1.8. *Find a convergent subsequence of* $x_n = (-1)^n$

Definition 4.1.3 (**Cauchy Sequence**)**.** *A sequence of real numbers* x_n *is said to be a Cauchy sequence if for every* $\epsilon > 0$ *there exists a natural number* $K := K(\epsilon)$ *such that* $|x_m - x_n| < \epsilon$ *for all* m, $n \geq K$.

Theorem 4.1.9 (**Cauchy Convergence Criterion**)**.** *A sequence of real numbers is convergent if and only if it is Cauchy sequence.*

Example 4.1.9. $x_n = \frac{1}{n}$ *is a Cauchy sequence.*

4.2 Sequence of Functions

What we have studied so far is the **sequence of constants** i.e., sequence of constant functions. In this section we shall discuss about the sequence of arbitrary functions. Therefore, the **sequence of function** is a sequence whose terms are *functions* rather than real numbers. Sequences of functions arise naturally in real analysis and are especially useful in obtaining approximations to a given function and defining new functions from known ones.

Let's begin with an example:

Example 4.2.1. *Consider the sequence of functions $f_n : A = [0, 1] \to \mathbb{R}$ defined by $f_n(x) := x^n$. We have $f_n(0) = 0$ for al n and $f_n(X) \to 0$ if $x < 1$, but $f_n(1) = 1$ for all n. Thus, $f_n(x)$ converges to a function $f(x)$ where*

$$f(x) := \begin{cases} 0 & \text{for } \ 0 \leq x < 1 \\ 1 & \text{if } \ x = 0, \end{cases}$$

Note that the limit of the sequence of functions depend on x that is the sequence of functions converges pointwise. Also note that the functions in the sequence are continuous but the limit function is not.

Now let's go through the following example:

Example 4.2.2. *Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) := \frac{\sin x}{n}$. We have $f(x) := \lim_{n\to\infty} \frac{\sin x}{n} = (\sin x) \lim_{n\to\infty} \frac{1}{n} = (\sin x) \cdot 0 = 0$ for all $x \in \mathbb{R}$.*

Here note that the limit of the sequence of functions does not depend on x that is the sequence of functions converges uniformly. Also note that the functions in the sequence are continuous and so does the limit function.

Therefore according to these examples the following two definitions are in order:

Definition 4.2.1 (**Pointwise Convergence**). *Let $< f_n >$ be a sequence of functions on $B \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A \subseteq B$, and $f : A \subseteq B \to \mathbb{R}$. Then the sequence $< f_n >$ converges pointwise on A to f if, for EACH $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$.*

In other words, $f_n \to f$ (pointwise) on A if for every $\epsilon > 0$ and for each $x \in A$, there exists a natural number $K := K(\epsilon, x)$ such that if $n \geq K$

$$|f_n(x) - f(x)| < \epsilon.$$

Definition 4.2.2 (**Uniform Convergence**). *Let $< f_n >$ be a sequence of functions on $B \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A \subseteq B$, and $f : A \subseteq B \to \mathbb{R}$. Then the sequence $< f_n >$ converges uniformly on A to f if, for ALL $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$.*

In other words, $f_n \to f$(uniformly) on A if for every $\epsilon > 0$ and for all $x \in A$, there exists a natural number $K := K(\epsilon)$ (depending on ϵ but not on x) such that if $n \geq K$

$$|f_n(x) - f(x)| < \epsilon.$$

Notations:

- $f_n \to f$(pointwise) on $A \Rightarrow f_n \to f$ on A or $f_n(x) \to f(x)$ for $x \in A$
- $f_n \to f$(uniformly) on $A \Rightarrow f_n \rightrightarrows f$ on A or $f_n(x) \rightrightarrows f(x)$ for $x \in A$

Example 4.2.3. *Discuss the uniform convergence of the sequence $< f_n >$ where $f : A = [0,1] \to \mathbb{R}$ is defined by $f_n(x) = \frac{x}{n}$, $n \in \mathbb{N}$.*
Solution: First find the pointwise convergence and then the uniform convergence. Clearly, $f_n(x) \to f = 0$ on A. We claim that this convergence is uniform as well. To this end, let $\epsilon > 0$, then $1/\epsilon > 0$, by Archimedean theorem, $\exists K \in \mathbb{N}$ such that $1/\epsilon < K \Rightarrow 1/K < \epsilon$. Now, if $n \geq K$, then

$$\begin{aligned} \Big|f_n(x) - f(x)\Big| &= \Big|\frac{x}{n} - 0\Big| \\ &= \frac{x}{n} \\ &\leq \frac{1}{n} \qquad \textit{for all } x \in A \\ &< \frac{1}{K} \\ &< \epsilon \end{aligned}$$

Therefore, $f_n \rightrightarrows f$ on A

Example 4.2.4. *Discuss the uniform convergence of the sequence $< f_n >$ where $f : A = [0,1] \to \mathbb{R}$ is defined by $f_n(x) = 1 - \frac{x}{n}$, $n \in \mathbb{N}$.*
Solution: First find the pointwise convergence and then the uniform convergence. Clearly, $f_n(x) \to f = 1$ on A. We claim that this convergence is uniform as well. To this end, let $\epsilon > 0$, then $1/\epsilon > 0$, by Archimedean theorem, $\exists K \in \mathbb{N}$ such that $1/\epsilon < K \Rightarrow 1/K < \epsilon$. Now, if $n \geq K$, then

$$\begin{aligned} \Big|f_n(x) - f(x)\Big| &= \Big|1 - \frac{x}{n} - 1\Big| \\ &= \frac{x}{n} \\ &\leq \frac{1}{n} \qquad \textit{for all } x \in A \\ &< \frac{1}{K} < \epsilon \end{aligned}$$

Therefore, $f_n \rightrightarrows f$ on A

Example 4.2.5. *Discuss the uniform convergence of the sequence* $< f_n >$ *where* $f : A = [0, 1] \to \infty$ *is defined by* $f_n(x) = \frac{x}{1+nx}$, $n \in \mathbb{N}$.
Solution: First find the pointwise convergence and then the uniform convergence. Clearly, $f_n(x) \to f = 0$ *on* A. *We claim that this convergence is uniform as well. To this end, let* $\epsilon > 0$, *then* $1/\epsilon > 0$, *by Archimedean theorem,* $\exists K \in \mathbb{N}$ *such that* $1/\epsilon < K \Rightarrow 1/K < \epsilon$. *Now, if* $n \geq K$, *then*

$$\Big|f_n(x) - f(x)\Big| = \Big|\frac{x}{1+nx} - 0\Big| < \Big|\frac{x}{nx}\Big| = 1/n \qquad \text{for all } x \in A$$
$$< 1/K < \epsilon$$

Therefore, $f_n \rightrightarrows f$ *on* A

Lemma 4.2.1. *A sequence of functions* f_n *on* $B \subseteq \mathbb{R}$ *does not converge uniformly on* $A \subseteq B$ *to a function* $f : A \to \mathbb{R}$ *iff for some* $\epsilon_0 > 0$ *there is a subsequence* f_{n_k} *of* f_n *and a sequence* x_k *in* A *such that*

$$\Big|f_{n_k}(x_k) - f(x_k)\Big| \geq \epsilon_0 \qquad \text{for all } k \in \mathbb{N}.$$

Example 4.2.6. *Discuss the uniform convergence of the sequence* $< f_n >$ *where* $f_n : A = [0, \infty) \to \mathbb{R}$ *is defined by* $f_n(x) = \frac{nx}{1+n^2x^2}$, $n \in \mathbb{N}$.
Solution: First find the pointwise convergence and then the uniform convergence. Clearly, $f_n(x) \to f = 0$ *on* A. *We claim that this convergence is not uniform on* A. *To this end, let* $\epsilon > 0$, *and let* $n_k = n$, $x_k = 1/n$, *then*

$$\Big|f_{n_k}(x_k) - f(x_k)\Big| = \Big|f_n(1/n) - f(1/n)\Big| = 1/2 \geq \epsilon \qquad \text{for all } n \in \mathbb{N}.$$

Therefore, f_n *is not uniformly convergent to* f *on* A

Theorem 4.2.2 (**Interchange of Limit and Continuity**)**.** *Let* $< f_n >$ *be a sequence of functions on a set* $A \subseteq \mathbb{R}$ *and suppose that* f_n *converges uniformly on* A *to a function* $f : A \to \mathbb{R}$. *Then* f *is continuous on* A.

Theorem 4.2.3 (**Interchange of Limit and Derivative**)**.** *(J. R. Marsden [4]) Let* $< f_n >$ *be a sequence of differentiable functions on a set* $A = (a, b) \subseteq \mathbb{R}$ *converging pointwise to* f *on* A. *Suppose that the derivatives* $< f_n' >$ *are continuous and converge uniformly to a function* g. *Then* f *is differentiable and* $f' = g$, *i.e.,*

$$\lim_{n\to\infty} f_n' = \big(\lim_{n\to\infty} f_n\big)'$$

Theorem 4.2.4 (**Interchange of Limit and Integral**)**.** *Let* $< f_n >$ *be a sequence of functions that are (Riemann) integrable on* $A = [a, b]$ *and suppose that* $< f_n >$ *converges uniformly on* A *to a function* f. *Then* f *is (Riemann) integrable on* A *and* $\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$ *i.e.,*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

.

Theorem 4.2.5 (**Bounded Convergence Theorem**). *Let* $< f_n >$ *be a sequence of (Riemann) integrable functions on* $A = [a, b]$ *and suppose that* $< f_n >$ *converges on* A *to a (Riemann) integrable function* f. *Suppose also that there exits* $M > 0$ *such that* $|f_n(x)| \leq M$ *for all* $x \in A$. *Then*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

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Chapter 5

Series and Series of Functions

5.1 Series

Definition 5.1.1 (**Series**). *Sum of the terms of an infinite sequence is called a series. Given a series* $\sum_{n=1}^{\infty} x_n = \sum x_n = x_1 + x_2 + x_3 + \cdots$, *let* s_n *denote its* n*-th partial sum:*

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + x_3 + \cdots + x_n.$$

If the sequence s_n *is convergent, i.e. if* x *is a real number such that* $\lim(s_n) = x$, *then the series* $\sum x_n$ *is called* **convergent** *and we write*

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k = x_1 + x_2 + x_3 + \cdots = x.$$

The number x *is called the* **sum** *of the series. Otherwise, the series is* **divergent**.

Note that $\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k$.

Theorem 5.1.1. *The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, *the series is divergent.*

Example 5.1.1. *Test whether the series* $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ *is convergent or divergent.*
Solution: $\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} 4(4/3)^{n-1}$ *is a geometric series with* $a = 4$ *and* $r = 4/3 > 1$. *So, the series is divergent.*

Theorem 5.1.2. *The p-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and divergent if* $p \leq 1$. *When* $p = 1$ *the series is called the* **harmonic** *series.*

Theorem 5.1.3. *If the series* $\sum_{n=1}^{\infty} x_n$ *is convergent , then* $\lim(x_n) = 0$. *But the converse is not true in general, e.g., harmonic series.*

Theorem 5.1.4 (The Test for Divergence). *If* $x_n \to \infty$ *or* $x_n \nrightarrow 0$ *then the series* $\sum_{n=1}^{\infty} x_n$ *is divergent.*

Example 5.1.2. *Test whether the series* $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ *is convergent or divergent. Solution: Here* $x_n = \frac{n^2}{5n^2+4} \to 1/5 \neq 0$ *so The Test for Divergence implies that the series is divergent.*

Theorem 5.1.5. *The sum, difference, and scalar multiple of two convergent series are convergent.*

Example 5.1.3. *Find the sum of the series* $\sum_{n=1}^{\infty}(\frac{3}{n(n+1)} + \frac{1}{2^n})$.
Solution: Here second series is a geometric series with $a = 1/2$ *and* $r = 1/2$, *so* $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = 1$. *The first series is* $\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} = \lim_{n\to\infty}(1-\frac{1}{n+1}) = 3 \cdot 1 = 3$. *Therefore the sum of the given series is* $3 + 1 = 4$.

Theorem 5.1.6 (**Cauchy Criterion for Series**). *A series* $\sum x_n$ *in* $\mathbb{R}$ *is convergent iff for each* $\epsilon > 0$ *there is a natural number* $K := K(\epsilon)$ *such that if* $m > n \geq K$, *then*

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \epsilon.$$

Definition 5.1.2. *We say that a series* $\sum x_n$ *is* **absolutely convergent** *if the series* $\sum |x_n|$ *is convergent in* $\mathbb{R}$. *A series is said to be* **conditionally convergent** *if it is convergent but not absolutely convergent.*

Theorem 5.1.7. *If a series is absolutely convergent, then it is convergent.*

5.1.1 Tests for Absolute Convergence

Theorem 5.1.8 (**Comparison Test**). *Let* x_n *and* y_n *be real sequences such that for some natural number* K,

$$0 \leq x_n \leq y_n \qquad \text{for } n \geq K.$$

Then the convergence of $\sum y_n$ *implies the convergence of* $\sum x_n$ *and the divergence of* $\sum x_n$ *implies the divergence of* $\sum y_n$.

Example 5.1.4. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$.
Solution: Note that $\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$ *by the p-series* $\sum \frac{1}{n^2}$ *converges and hence by the Comparison Test the given series is convergent.*

Theorem 5.1.9 (**Limit Comparison Test**). *Let* x_n *and* y_n *be positive real sequences and* $L = \lim(x_n/y_n)$
(a) If $L \neq 0$, *then* $\sum x_n$ *is convergent iff* $\sum y_n$ *is convergent.*
(b) If $L = 0$ *and* $\sum y_n$ *is convergent, then* $\sum x_n$ *is convergent.*

Example 5.1.5. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$.
Solution: Note that the dominant part of the numerator is $2n^2$ *and the dominant part of the denominator is* $\sqrt{n^5}$. *This suggests taking* $x_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $y_n = \frac{2n^2}{\sqrt{n^5}} = \frac{2}{n^{1/2}}$, $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{1/2}}{2} = 1$ *Since* $\sum y_n = 2\sum 1/n^{1/2}$ *is divergent (p-series with* $p = 1/2 < 1$*), the given series diverges by the Limit Comparison Test.*

Theorem 5.1.10 (**Root Test**)**.** *Let* x_n *be a sequence in* $\mathbb{R}$ *and*

$$r := \lim(|x_n|^{1/n}).$$

Then $\sum x_n$ *is absolutely convergent if* $r < 1$ *and divergent if* $r > 1$.

Example 5.1.6. *Test the series* $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$
Solution: Root Test with $x_n = \left(\frac{2n+3}{3n+2}\right)^n$ *gives:* $\sqrt[n]{|x_n|} = \frac{2n+3}{3n+2} \to 2/3 < 1$ *Thus, the given series converges by the Root Test.*

Theorem 5.1.11 (**Ratio Test**)**.** *Let* x_n *be a sequence in* $\mathbb{R}$ *and*

$$r := lim(|x_{n+1}|/|x_n|).$$

Then $\sum x_n$ *is absolutely convergent if* $r < 1$ *and divergent if* $r > 1$.

Example 5.1.7. *Test the series* $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ *for absolutely convergence.*
Solution: Ratio Test with $x_n = (-1)^n \frac{n^3}{3^n}$ *gives:* $|\frac{a_{n+1}}{a_n}| = |\frac{1}{3}(\frac{n+1}{n})^3| = |\frac{1}{3}(1+1/n)^3| \to \frac{1}{3} < 1$ *Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.*

Theorem 5.1.12 (**Raabe's Test**)**.** *Let* x_n *be a sequence of nonzero real numbers and*

$$r := lim\Big(n\Big(1 - \frac{|x_{n+1}|}{|x_n|}\Big)\Big)$$

. *Then* $\sum x_n$ *is absolutely convergent if* $r > 1$ *and is not absolutely convergent if* $r < 1$.

Theorem 5.1.13 (**Integral Test**)**.** *Let* f *be a continuous, positive decreasing function on* $[1, \infty)$ *and* $x_n = f(n)$. *Then if* $\int_1^{\infty} f(x)dx$ *is convergent (divergent), then* $\sum x_n$ *is convergent (divergent).*

Example 5.1.8. *Test the series* $\sum_{n=1}^{\infty} \frac{lnn}{n}$ *for convergence.*
Solution: The function $f(x) = (lnx)/x$ *is positive and continuous for* $x > 1$ *because the logarithm function is continuous. But it is not clear whether or not* f *is decreasing, so* $f'(x) = \frac{1-lnx}{x^2}$. *Thus,* $f'(x) < 0$ *when* $1 - lnx < 0$, *i.e.,* $x > e$. *So* f *is decreasing when* $x > e$. *Here* $\int_1^{\infty} \frac{lnx}{x}dx = \lim_{t\to\infty} \int_1^t \frac{lnx}{x}dx = \infty$ *Therefore, by the Integral Test the given series is divergent.*

5.1.2 Tests for Nonabsolute Convergence

Definition 5.1.3 (**Alternating Series**)**.** *An alternating series is a series whose terms are alternately positive and negative. The n-th term of an alternating series is defined by* $x_n = (-1)^{n-1}y_n$, $x_n = (-1)^n y_n$, $x_n = (-1)^{n+1}y_n$ *where* $n \in \mathbb{N}$ *and* $y_n > 0$

Theorem 5.1.14 (**Alternating Series Test**)**.** *Let* x_n *be a decreasing sequence of positive real numbers with* $\lim(x_n) = 0$. *Then the alternating series* $\sum(-1)^{n+1}x_n$ *is convergent.*

Example 5.1.9. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$.

5.2 Series of Functions

We have studied the series of constants(i.e. constant functions). He we shall study the series of functions rather than constants.

Definition 5.2.1 (**Pointwise and Uniform Convergence of Series**). *Let the function f and the sequence of functions $< f_n >$ be defined on $B \subseteq \mathbb{R}$ and $s_n = \sum_{k=1}^{n} f_k$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$ in $B \subseteq \mathbb{R}$.*

Then if $s_n \to f$(pointwise) on B (i.e., $s_n \to f$ on B), then $\sum_{n=1}^{\infty} f_n(x) \to f(x)$ for $x \in B$ and we write $\sum_{n=1}^{\infty} f_n(x) = f(x)$(pointwise)

Then if $s_n \to f$(uniformly) on B (i.e., $s_n \rightrightarrows f$ on B), then $\sum_{n=1}^{\infty} f_n(x) \rightrightarrows f(x)$ for $x \in B$ and we write $\sum_{n=1}^{\infty} f_n(x) = f(x)(uniformly)$

Theorem 5.2.1 (**Interchange of Limit and Sum**). *Let $< f_n >$ be a sequence of functions on a set $A \subseteq \mathbb{R}$ and suppose that $\sum f_n$ converges uniformly on A to a function $f : A \to \mathbb{R}$. Then f is continuous on A and*

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x).$$

Theorem 5.2.2 (**Interchange of Sum and Derivative**). *(J. R. Marsden [4]) If $< f_n >$ are differentiable functions on a set $A = (a, b) \subseteq \mathbb{R}$, the $< f_n' >$ are continuous, $\sum_{n=1}^{\infty} f_n$ converge pointwise, and $\sum_{n=1}^{\infty} f_n'$ converges uniformly, then*

$$\Big(\sum_{n=1}^{\infty} f_n\Big)' = \sum_{n=1}^{\infty} f_n'.$$

Theorem 5.2.3 (**Interchange of Integral and Sum**). *Let $< f_n >$ be a sequence of functions that are (Riemann) integrable on $A = [a, b]$ and suppose that $< f_n >$ converges uniformly on AThen we may interchange the order of integration and summation:*

$$\int_a^b \Big(\sum_{n=1}^{\infty} f_n(x)\Big)dx = \sum_{n=1}^{\infty} \Big(\int_a^b f_n(x)dx\Big).$$

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Chapter 6

Limit and Continuity of a Function

6.1 Limit of a Function

Definition 6.1.1 (**Cluster Point or Accumulation Point**). *Let $A \subseteq ReE$. A point $c \in \mathbb{R}$ is called a cluster point of A if every $\delta-$neighborhood $V_\delta(c) := (c-\delta, c+\delta)$ of c contains at least one point of A other than c, i.e.,*

$$V_\delta(c) - \{c\} \cap A \neq \phi$$

Definition 6.1.2 (**Limit Point**). *Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is called a limit point of A if every $\delta-$neighborhood $V_\delta(c) := (c-\delta, c+\delta)$ of c contains at least one point of A, i.e.,*

$$V_\delta(c) \cap A \neq \phi$$

Example 6.1.1.

(a) Let $A = (0,1) \cup \{3\}$. Then the set of cluster points of A is $[0,1]$ and the set of limit points of A is $[0,1] \cup \{3\}$.

(b) A finite set has no cluster points.

(c) $\mathbb{N}$ has no cluster points.

(d) $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ has only one cluster point 0.

(e) Let $S = \{x \in \mathbb{R} : x \in [0,1]$ and x is rational$\}$. Then all points of $[0,1]$ are accumulation points of S.

Definition 6.1.3 ($\epsilon - \delta$: **Definition of Limit of a Function**). *Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. We say that a real number L* **is a limit of** *f* **at** *c, written $\lim_{x \to c} f(x) = L$ or $f(x) \to L \quad as \quad x \to c$ if, given any $\epsilon-$neighborhood $V_\epsilon(L)$ of L, there exists a $\delta-$neighborhood $V_\delta(c)$ of c such that if $x \neq c$ is any point of $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\epsilon(L)$.*

In other words, L **is a limit of** *f* **at** *c, written $\lim_{x \to c} f(x) = L$ or $f(x) \to L \quad as \quad x \to c$, if given $\epsilon > 0$ there exists a $\delta := \delta(\epsilon) > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$*

Example 6.1.2. *Apply $\epsilon - \delta$ definition of limit to illustrate that*

$$\lim_{x \to 4}(2x-2) = 6$$

Solution: Here $f(x) = 2x - 2$, $c = 4$, and $L = 6$.
Let $\epsilon > 0$. WANT to find a $\delta := \delta(\epsilon) > 0$ such that
if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$ i.e.,
if $0 < |x - 4| < \delta$, then $|f(x) - L| < \epsilon$
Now, $|f(x) - L| = |(2x - 2) - 6| = 2|x - 4| < 2\delta$
If we choose $\delta = \frac{\epsilon}{2} =: \delta(\epsilon)$, then
if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$
Therefore, $\lim_{x \to c} f(x) = L \Rightarrow \lim_{x \to 4}(2x - 2) = 6$

Theorem 6.1.1 (**Sequential Criterion**). *Let $f : A \to \mathbb{R}$ and c be a cluster point of A; then:*
(i) $\lim_{x \to c} f(x) = L$ if and only if
(ii) for every sequence x_n in A that converges to c such that $x \neq c$ for all $n \in \mathbb{N}$, the sequence $f(x_n)$ converges to L.

Theorem 6.1.2 (**Divergence Criterion**). *Let $f : A \to \mathbb{R}$ and c be a cluster point of A; then:*
(i) The function f does not have a limit at c
if and only if
(ii) $\exists$ a sequence $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence x_n converges to c but the sequence $f(x_n)$ does not converges in $\mathbb{R}$

Example 6.1.3. *Show that $\lim_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.*

Solution: Here let $f(x) = \frac{1}{x}$ for $x > 0$. Take the sequence $x_n := 1/n$ for $n \in \mathbb{N}$, then $f(x_n) = n$ does not converge in $\mathbb{R}$. Hence by the Divergence Criterion the given sequence does not exist in $\mathbb{R}$.

6.2 Continuous Function

Definition 6.2.1 (**Continuous Function**). *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is continuous at a point $c \in A$, written $\lim_{x \to c} f(x) = f(c)$ if, given any $\epsilon-$neighborhood $V_\epsilon(f(c))$ of $f(c)$, there exists a $\delta-$neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c) \cap A$, then $f(x) \in V_\epsilon(f(c))$.*

In other words, f is continuous at a point $c \in A$, written $\lim_{x \to c} f(x) = f(c)$ if, given any $\epsilon > 0$ there exists a $\delta := \delta(\epsilon, x) > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$

Theorem 6.2.1 (**Discontinuity Criterion**). *Let $f : A \to \mathbb{R}$ and $c \in A$; then:*
(i) The function f is discontinuous at c
if and only if
(ii) $\exists$ a sequence x_n in A such that the sequence x_n converges to c, but the sequence $f(x_n)$ does not converges to $f(c)$.

Example 6.2.1.

(a) $f(x) = x$ *is continuous on* $\mathbb{R}$.

(b) $f(x) = x^2$ *is continuous on* $\mathbb{R}$.

(c) $f(x) = 1/x$ *is continuous on* $A = (0, \infty)$.

(d) $f(x) = 1/x$ *is not continuous at* 0.

Theorem 6.2.2 (**Combination of Continuous Functions**). *The sum, difference, product, scalar multiple, and division (***if the denominator is not zero***) of two continuous functions are continuous.*

Theorem 6.2.3 (**Polynomial functions**). *are always continuous.*

Definition 6.2.2 (**Bounded Function**). *A function* $f : A \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be bounded on* A *if there exists a constant* $M > 0$ *such that* $|f(x)| \leq M$ *for all* $x \in A$.

Example 6.2.2.

(a) $f(x) = x$ *is bounded on* $A = [-7, 2]$ *but unbounded on* $\mathbb{R}$.

(b) $f(x) = 1/x$ *is continuous on* $A = (0, \infty)$ *but not bounded on* A. $f(x)$ *is not even bounded when restricted to the set* $B = (0, 1)$.

Theorem 6.2.4 (**Boundedness Theorem**). *Let* $I := [a, b]$ *be a closed bounded interval and let* $f : I \to \mathbb{R}$ *be continuous on* I. *Then* f *is bounded on* I.

Definition 6.2.3 (**Absolute Extremum of a Function**). *Let* $f : A \subseteq \mathbb{R} \to \mathbb{R}$ *be a function. We say that* f *has an absolute maximum on* A *if there is a point* $x^\star \in A$ *such that*

$$f(x^\star) \geq f(x) \qquad \text{for all} \qquad x \in A.$$

We say that f *has an absolute minimum on* A *if there is a point* $x_\star \in A$ *such that*

$$f(x_\star) \leq f(x) \qquad \text{for all} \qquad x \in A.$$

We say that $x^\star$ *is an absolute maximum point for* f *on* A, *and that* $x_\star$ *is an absolute minimum point for* f *on* A, *if they exist.*

Example 6.2.3.

1. *Continuous function on a set* A *does not necessarily have an absolute maximum or an absolute minimum on the set.*
2. $f(x) = 1/x$ *has neither an absolute maximum nor an absolute minimum on the set* $A = (0, \infty)$.
3. *The same function has neither an absolute maximum nor an absolute minimum when restricted to the open set* $B = (0, 1)$.
4. *While the same function has BOTH an absolute maximum and an absolute minimum when restricted to the closed set* $C = [0, 1]$.

5. *The function $f(x) = x^2$ on $A = [-1, +1]$ has two points $x = \pm 1$ giving the absolute maximum and a single point $x = 0$ yielding the absolute minimum on A.*

6. *The constant function $f(x) = c$ is such that for $x \in \mathbb{R}$ is such that EVERY point of $\mathbb{R}$ is BOTH an absolute maximum and an absolute minimum point for f.*

Theorem 6.2.5 (**Maximum-Minimum(Maximin) Theorem**)**.** *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.*

Theorem 6.2.6 (**(Bolzano's) Intermediate Value Theorem**)**.** *Let I be any interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a,\ b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in \mathbb{I}$ between a and b such that $f(c) = k$.*

Corollary 6.2.7. *Let $I = [a, b]$ be a closed, bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying*

$$\inf f(I) \leq k \leq \sup f(I),$$

then there exists a number $c \in I$ such that $f(c) = k$.

The following corollary provides the Location of Roots:

Corollary 6.2.8. *Let I be any interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a < b$ are numbers in I such that $f(a) < 0 < f(b)$(or such that $f(a) > 0 > f(b)$), then there exists a point $c \in (a, b)$ such that $f(c) = 0$.*

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Chapter 7

Differentiation

Definition 7.0.4 (**The Derivative**). *Let $f : I \to \mathbb{R}$ where $c \in I$, an interval. We say that a real number L is the* **derivative of** f **at** c *if for any given number $\epsilon > 0$ there exists a number $\delta(\epsilon) > 0$, then*

$$\left|\frac{f(x) - f(c)}{x - c} - L\right| < \epsilon.$$

In this case we say that f is **differentiable at** c**,** *and we write $f'(c)$ for L.*

In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Theorem 7.0.9. *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.*

Is the converse true? **WHY?**

Theorem 7.0.10. *The sum, difference, scalar multiple, product, and division(not always) of two differentiable functions are differentiable.*

Theorem 7.0.11 (**Rolle's Theorem**). *Let $f : I = [a, b] \to \mathbb{R}$ be continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.*

The Mean Value Theorem relates the values of a function to values of its derivative which is stated as follows.

Theorem 7.0.12 (**Mean Value Theorem**). *Let $f : I = [a, b] \to \mathbb{R}$ be continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists at least one point c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

One can easily deduce Roll's Theorem from Mean Value Theorem. Now we shall present the **Taylor's Theorem** which is simply the generalization of the **Mean Value Theorem**.

Theorem 7.0.13 (**Taylor's Theorem**). *Let $f : I = [a, b] \to \mathbb{R}$ be a function such that f and its derivatives up to order n are continuous on a closed interval $[a, b]$ and $f^{(n+1)}$ exists on the open interval (a, b). If $x_0 \in I$, then for any x in I there exists a point c between x and x_0 such that*

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{(n+1)}$$

where the last term

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{(n+1)}$$

is called the remainder.

Theorem 7.0.14 (**Taylor's Inequality or The Remainder Estimation Theorem**). *If $|f^{(n+1)}(x)| \leq M$ for all $x \in I$, and a number $x_0 \in I$, then*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

Theorem 7.0.15. *If $n \to \infty$ then $R_n(x) \to 0$.*

Hence the Taylors Theorem gives us the Taylor's series:

Theorem 7.0.16 (**Taylor's Series**).

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

Taylor's Series evaluated at $x_0 = 0$ is called the **Maclaurin's series which is given as follows:**

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Example 7.0.4.

- *Expand $f(x) = e^x$.*
- *The function $f(x) = \sin x$ about the point $x_0 = \pi/2$.*

Theorem 7.0.17 (**L'Hospital's Rule**). *Suppose that f and g are differentiable and $g'(x) \neq 0$ near x_0 (except possibly at x_0). Suppose that $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$ or that $\lim_{x\to x_0} f(x) = \pm\infty$ and $\lim_{x\to x_0} g(x) = \pm\infty$ (In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.)*

Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

If the limit on the right side exists(or is ∞ or $-\infty$).

Example 7.0.5.

- *Find $\lim_{x\to 1} \frac{\ln x}{x-1}$*

- *Find* $\lim_{x\to 0} \frac{\tan x - x}{x^3}$
- *Find* $\lim_{x\to 2} \frac{x^2-4}{x-2}$
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Chapter 8

Riemann Integration

Riemann Integration

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Appendix A

Reviews

Reviews

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Bibliography

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