the fundamental group of a loop

why π(● : 𝕊) ≅ ℤ

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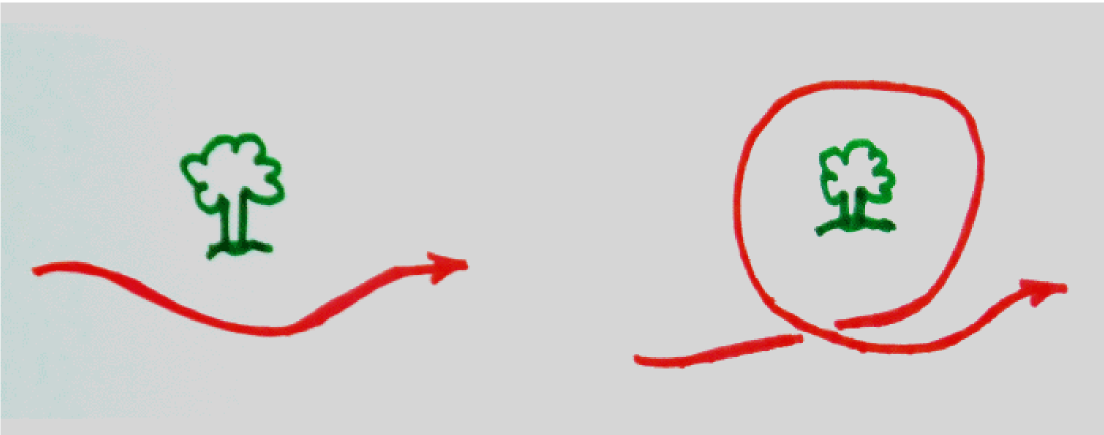
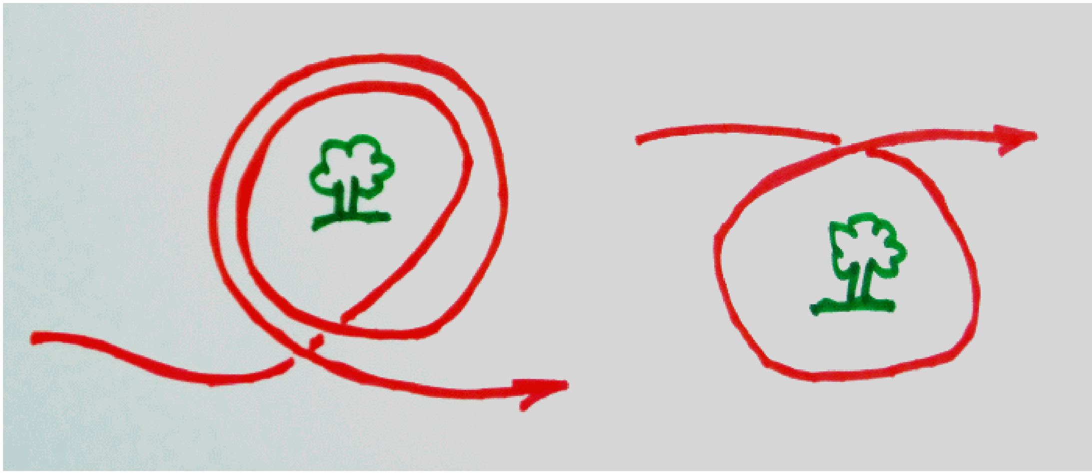
0: ropes and trees

You are traversing a strip of land trailing a rope behind you. You come across a tree. How many ways can you continue your journey past the tree? You could just ignore it and pass on the left (or on the right, for that matter – you can pick either chirality but you must pick one and stick to it). Or you could head past the tree, but steer left as you reach it and wind round it once before heading on.



Or you could do the same, but loop round twice before heading on. Or you could loop around once in the opposite direction, ie pass with the tree on your right (ie make the other chiral choice, the mirror image):



Note that we do not care exactly how we wind round the tree – we assume our rope is as elastic or as rigid as we wish, and that pulling the rope tight or loosening it does not fundamentally change the nature of the loop in question.

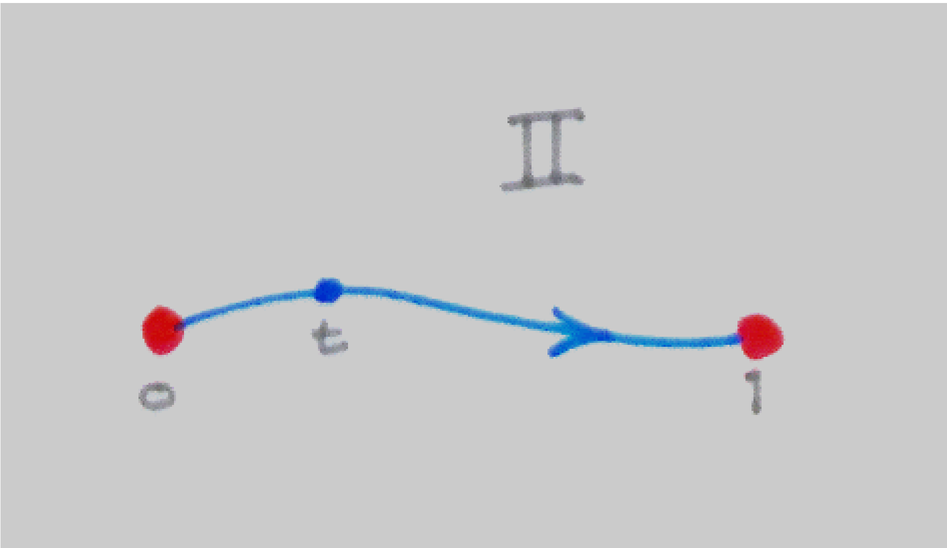
Intuitively it seems obvious that the four different journeys outlined above represent four different classes of journeys, ones that cannot be transformed into another by tightening or loosening the rope in various ways. (We assume we cannot dig beneath tree's roots or lift the rope over it, nor can we cut the tree down or the rope in two, nor can we pass the rope through the trunk or through itself by some other means.)

Furthermore, the various classes of paths correspond to the integers: there is a zero path (walk straight past), a unit path (loop round once widdershins), multiples thereof (loops twice, or thrice), a negative unit (loop once sunwise), &c. These follow a primitive additive arithmetic: tracing out one loop followed by another is equivalent to adding them, with the zero, unit and negative playing their usual roles.

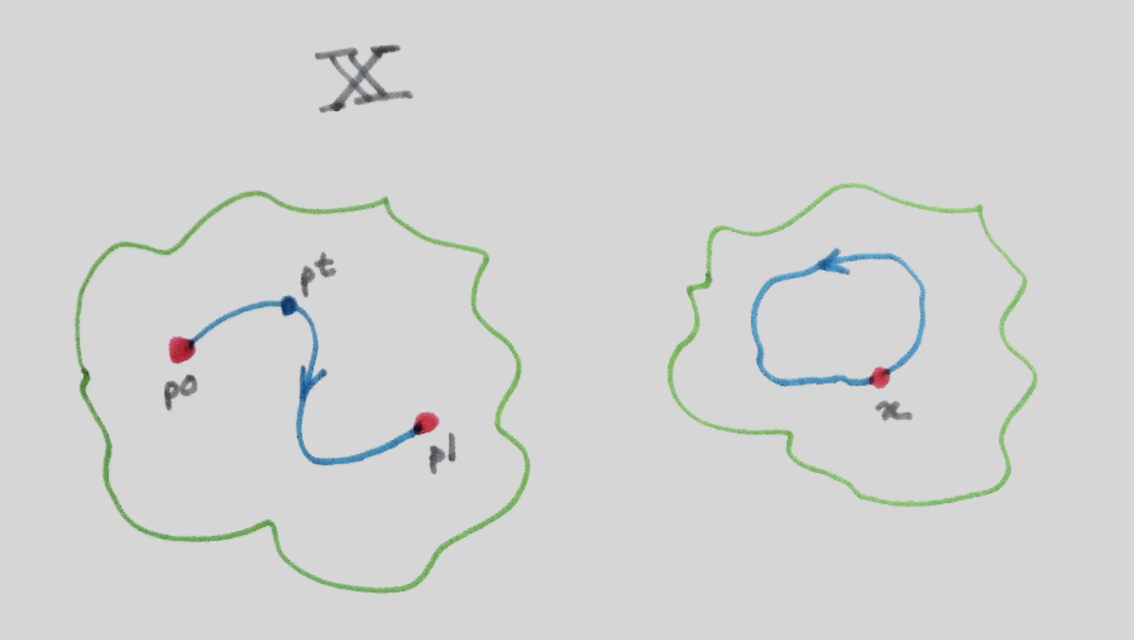
This fact was first formalised by the great French mathematician, engineer and philosopher Henri Poincaré in the 1890s. He invented the terminology we use today: we say the fundamental group of a loop is isomorphic to the additive group of integers, ie it is freely generated by iterating and inverting the unit loop. In short, every loop is the same as $n \otimes \tau$ where n is an integer (a widget of type $\mathbb{Z}$) and τ is the unit loop described above. The four loops above, respectively, are: $0 \otimes \tau$, $\tau = 1 \otimes \tau$, $2 \otimes \tau$, $-\tau = -1 \otimes \tau$

1: paths and loops

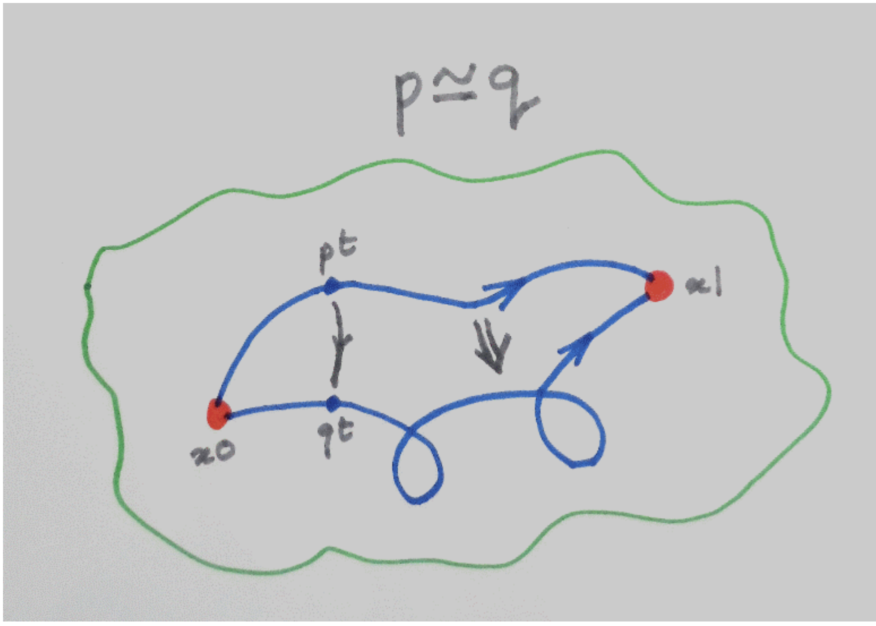
We start with the $\mathbb{I}$, the unit closed real **interval**. We can think of this as a straight line from 0 to 1 of length 1. We typically denote a widget of type $\mathbb{I}$ as t, and think of it as an instant of our journey, ie a point in time between 0, our departure point or source, and 1 our arrival point or target. We flow continuously through $\mathbb{I}$ from t = 0 to t = 1.



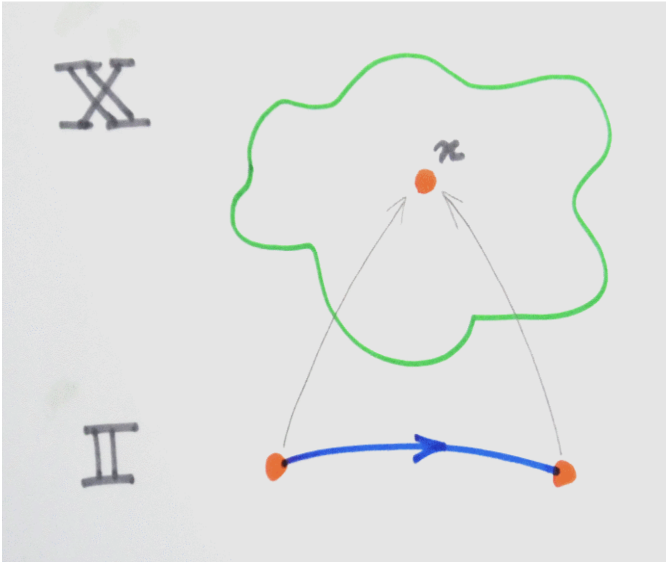
A **path** in X is a continuous map p from 𝕀 to any suitable space 𝕏 (we leave aside the delicate question of how what “suitable” means here; turns out almost any moderately well-behaved topological space will do (any compactly generated space, to be precise)). Every path has a source p0 and a target p1; we say p is a path from p0 to p1 in 𝕏. A path from x to itself in 𝕏 is called a **loop** at x.



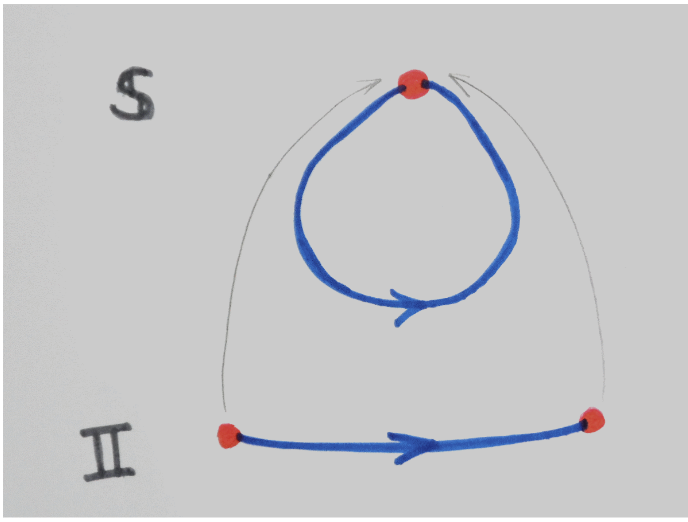
We do not care much about continuous alterations to paths that bend, stretch or collapse them. Any two such paths p and q from x0 to x1 in 𝕏 will be considered the **same**, or “homotopic”, if p can be continuously deformed into q in 𝕏 while keeping the endpoints x0 and x1 fixed. We write this p ≃ q and say p and q are “of the same homotopy class”, p and q are the same (though not necessarily equal as paths).



One simple path is one that starts from x in 𝕏 and stays there for the duration. This is the **constant** path at x. It simply collapses the interval 𝕀 to the point x in 𝕏. This path starts and ends at x, and thus is a loop at x, but not every such loop is so trivial.



The simplest nontrivial loop at a point can be constructed by identifying the endpoints of the interval. Our path now bends round itself in a particular direction and returns to its origin. This is the quotient map τ (or coequaliser of 0 and 1 in 𝕀) that sends 0 and 1 to 𝕀 to the basepoint • of a circular loop 𝕊. It winds around the circle once anticlockwise before returning to its origin • coming in from the opposite direction.



What Poincaré discovered – and what we will shortly prove using modern notation and concepts – was that the homotopy classes of loops at • in 𝕊 form a group, the free group generated by the unit τ, ie any loop at • in 𝕊 is the same as n⊗τ for a unique integer n. We call this group the **fundamental** group at • and write it π(• : 𝕊). It is isomorphic as a group to ℤ, the additive group of integers.

next: the inverse image chain