Writing Assignment #2

Proposition. Let $H \leq G$ and define a relation on G by $x \sim y$ if $y^{-1}x \in H$, then $\sim$ is an equivalence relation on G and the equivalence classes of $\sim$ are left cosets of H in G.

To show that $\sim$ is an equivalence relation on G we will check to verify that $\sim$ is reflexive, symmetric and transitive

Reflexive: Suppose that $x \in G$ then $x^{-1}x \in H$, since H is a subgroup the identity is in H, so then $a \sim a$ and $\sim$ is reflexive.
Symmetric: Suppose that $x, y \in G$ and $x \sim y$ then $y^{-1}x \in H$. Since H is a subgroup, H is closed under inverses then $(y^{-1}x)^{-1} \in H$. So then $(y^{-1}x)^{-1} = x^{-1}(y^{-1})^{-1} = x^{-1}y \in H$ and then $y \sim x$ and $\sim$ is symmetric.
Transitive: Suppose that $x \sim y$ and $y \sim z$ then $y^{-1}x \in H$ and $z^{-1}y \in H$, since H is a subgroup it is closed under products so then,

$$(z^{-1}y)(y^{-1}x) = z^{-1}(yy^{-1})x = z^{-1}ex = z^{-1}x$$

and $z^{-1}x \in H$ and then $x \sim z$ and $\sim$ is transitive.

Thus $\sim$ satisfies the reflexive, symmetric and transitive properties therefore $\sim$ is an equivalence relation on G.

For any $x \in G$ the equivalence classes of x are $[x] = \{y \in G : y \sim x\}$ and the left cosets of H in G are $xH = \{xh : h \in H\}$, all that is left is to show that $[x] = xH$ and since these objects are sets we must show that $[x] \subseteq xH$ and $xH \subseteq [x]$.

Suppose that $a \in [x]$, so then $a \sim x$ and $x^{-1}a \in H$. If we set $h = x^{-1}a$ for some $h \in H$ then $a = xh$ and $a \in xH$. Therefore $[x] \subseteq xH$.

Next suppose that $a \in xH$ then set $a = xh$ for some $h \in H$, and then $x^{-1}a = h$ and $x^{-1}a \in H$. By definition of $\sim$, $a \sim x$ so then $a \in [x]$ and $xH \subseteq [x]$.

Finally since we have shown that $[x] \subseteq xH$ and $xH \subseteq [x]$, then $[x] = xH$ and the equivalence classes of $\sim$ are left cosets of H in G. $\blacksquare$