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$\iint$ DOULE INTEGRALS

Integration and its inverse, differentiation, are the two main operations of calculus. Integration is the evaluation of a function over a definite or indefinite integral. It is notated as:

Definite	**Indefinite**
$\int_a^b f(x)dx$	$\int f(x)dx$

And they are equivalent to:

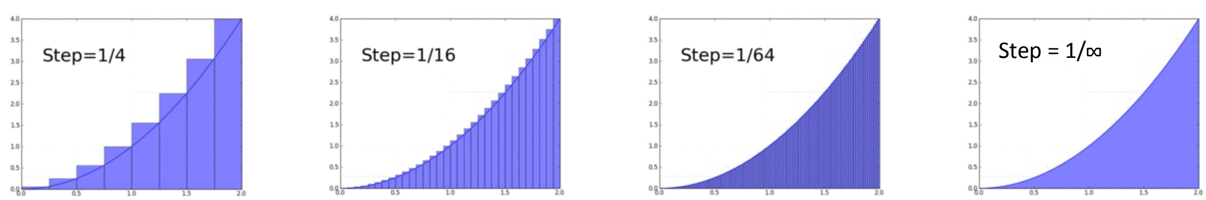
$$\int_a^b f(x)dx = F(b) - F(a) \qquad\qquad \int f(x)dx = F(x)$$

Early mathematicians thought of integration as an infinite sum of rectangles of immeasurably small width. Bernhard Riemann created the first mathematical definition of integrals. With rectangle width *Δx* and *n* number of rectangles, the integral can be approximated using Riemann sums. When *n* approaches infinite, the width of each individual rectangle approaches zero, or is infinitesimal. When Δx is infinitesimal, the Riemann sum is equal to the integral of the function:

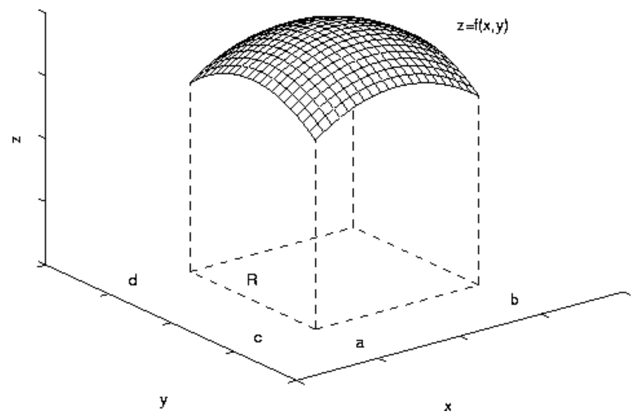
$$\int f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where x_i represents each rectangle

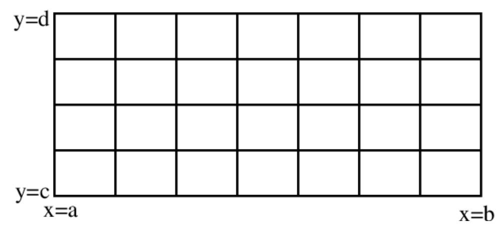
Essentially, the smaller Δx is, the closer the approximation is to the actual integral. As shown:



Functions with multiple variables can also be integrated using multiple integrals. In contrast to single variable functions, whose integrals represent the area underneath the function over an interval, multiple variable functions' integrals represent the volume underneath the function. Riemann sums can still be used to approximate multiple-variable integration. In three-dimensional Riemann sums, functions are approximated using rectangular prisms as opposed to two-dimensional rectangles. The base of these rectangular prisms are represented by $dxdy \text{ or } dA$, where *A* represents area. For definite integrals, the boundaries of the interval are defined by the region R, an area on a two-dimensional plane, as shown below:

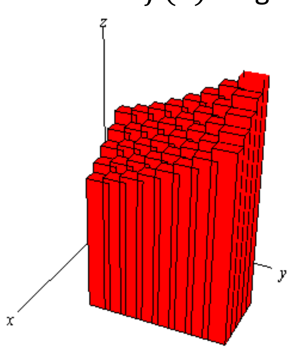


Similar to the step size, or Δx, in single-variable Riemann Sums, R can be split up into partitions that can be used to approximate the integral. These rectangles are represented as ΔA, and their height as $f(x_i, y_j)$. Therefore, their volumes are equal to $f(x_i, y_j)\Delta A$. The sum of the volumes of these rectangular prisms will approximately equal the volume under $f(x, y)$. It can be represented by:



$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta A$$

Unlike the Riemann sums for single-variable integrals, the rectangles for double integrals must be added up along x and y, whereas single-variable integrals only had rectangles of Δx width and $f(x)$ height. Because of this, the equation is a double summation.



In single-variable Riemann sums, the approximation equals the actual value of the integral when Δx approaches infinitesimal. In multiple-variable Riemann sums, the approximation equals the actual value of the integral when ΔA is infinitesimal:

$$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta A$$

Where the integral for $z = f(x, y)$ over the region R, is represented by:

$$\iint_R f(x, y)dA$$

This double integral equation represents the double variables of f(x,y). This equation can also be written as an iterated integral. With $R = [a,b] * [c,d]$, an iterated version of $f(x,y)$ splits up the integral into two definite integrals that can be easier to integrate:

$$\int_a^b \int_c^d f(x,y)dydx \qquad or \qquad \int_c^d \int_a^b f(x,y)dxdy$$

Both of these equations are essentially equal to each other, however, depending on the boundaries of R, one can be easier to solve than the other. They are both solved the same way:

$$\int_a^b \int_c^d f(x,y)dydx \qquad \int_c^d \int_a^b f(x,y)dxdy$$

$$\int_a^b [\int_c^d f(x,y)dy]dx \qquad \int_c^d [\int_a^b f(x,y)dx]dy$$

Separate the inside integral and solve it

$$\int_a^b [F(d) - F(c)]dx \qquad \int_c^d [F(b) - F(a)]dy$$

With c representing the result of inside integral, solve the outside integral

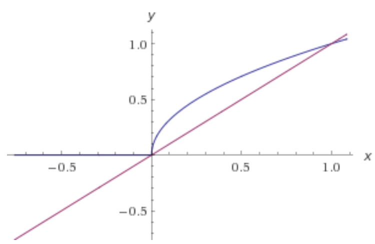
$$C(b) - C(a) \qquad C(d) - C(c)$$

Using this integral you can find the volume under the function f(x,y). Note that the boundaries need to be changed in order to change the boundary to be with respect to the other variable. This is very similar to taking the single-variable integral of a function with respect to x instead of y. For example:

$$\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y} dydx$$

$\frac{e^y}{y}$ Cannot be analytically integrated. Another method must be used

Set the boundaries of each integrand with the variable that it is in respect to. For this example, $y = \sqrt{x}$, $y = x$, $x = 1$, and $x = 0$ are the boundaries.



$$\int_0^1 \int_{y^2}^{y} \frac{e^y}{y} dxdy$$

Flip the order of the integrals and set the new boundaries based on the variable that the inside integral is in respect to. In this case, $y = \sqrt{x}$ becomes $y^2 = x$, and becomes the lower boundary. $\int \frac{e^y}{y} dx$can easily calculate to $\frac{e^y}{y} x + c$.