Math 2250 Lab 12 Name/Unid: ______________________________
Due Date:

1. Let

$$f(t) = \begin{cases} 0 & \text{if } 2n-2 \leq t < 2n-1 \\ 1 & \text{if } 2n-1 \leq t < 2n \end{cases}$$

where $n = 1, 2, 3, \ldots$.

(a) Sketch the graph of $f(t)$.

(b) Show that the function $f(t)$ can be written as $\sum_{m=1}^{\infty}(-1)^{m+1}u_m(t)$, where $u_m(t)$ is the step function at $t = m$ i.e.

$$u_m(t) = \begin{cases} 0 & \text{if } t < m \\ 1 & \text{if } m \leq t \end{cases} .$$

(c) Rewrite your answer to part (b) as a geometric series to obtain the result.

$$\mathscr{L}\{f(t)\} = \frac{e^{-s}}{s(1-e^{-s})}.$$

You must state why the series is geometric. The sum of a geometric series is given by:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{for } |r| < 1.$$

2. Consider a physical system in which the output or response $x(t)$ to an input function $f(t)$ is described by some differential equation of the form

$$ax'' + bx' + cx = f(t)$$

for unknown parameters a, b, c determined by the system. Assume that you only know that all inputs to this system $x(t)$ satisfy

$$x(t) = \int_0^t e^{-3\tau} \sin(4\tau) f(t-\tau) d\tau$$

for any forcing function $f(t)$ and that this system is passive initially; that is, $x(0) = x'(0) = 0$. This is an example of Duhamel's principle, which uses convolution to reduce the problem of finding a system's output for all possible inputs to a single inverse Laplace transform.

(a) $w(t) = e^{-3t} \sin(4t)$ is called the weight function of the system. Use this to find the transfer function $W(s)$, where $w(t) = \mathscr{L}^{-1}\{W(s)\}$.

(b) The transfer function, $W(s)$, is the ratio of output $x(t)$, to input, $f(t)$ but in the s–domain. That is

$$W(s) = \frac{X(s)}{F(s)},$$

where $X(s) = \mathscr{L}\{x(t)\}$ and $F(s) = \mathscr{L}\{f(t)\}$. Use these facts, and $W(s)$ found in part (a) to determine the left hand side of the nonhomogeneous ODE describing your mechanical/electrical system.

If this was a mass, spring, damper system, what would the values of m, c and k be? If it was an electrical system, what would the values of L, R and C be?

Hint: Do the Laplace transform method of solving ODEs in reverse. Your starting point is $X(s)/W(s) = F(s)$. Also, you were not given a specific function for $F(s)$, thus just assume $F(s) = \mathscr{L}\{f(t)\}$.

3. The transfer function in the previous question is also useful in the signal processing. We now consider a system with the input signal $f(t)$ and the output signal $y(t)$. The transfer function $W(s)$ is again the ratio

$$W(s) = \frac{Y(s)}{F(s)},$$

where $F(s) = \mathscr{L}\{f(t)\}$ and $Y(s) = \mathscr{L}\{y(t)\}$. It is given that the transfer function

$$W(s) = \frac{5e^{-\tau s}}{1 + Ts},$$

where $\tau = 0.2$ and $T = 2$. Suppose $f(t)$ is the constant unit function i.e. $f(t) \equiv 1$.

(a) Find the output signal $y(t)$. (Hints: first compute $Y(s)$ and then use the inverse Laplace transform to find $y(t)$.)

(b) Use either computer software or by hand to sketch the graph in (a).

(c) In part (b), you should notice that $y(t)$ is zero in the beginning. The time that the output signal $y(t)$ starts to increase is called the *time delay*. Find the time delay.

(d) Find the time that the output signal $y(t)$ reaches 63% of its steady value.

(e) How does the time delay change if the value of τ is increased? How does the time in part(d) change if the value of T is increased?

4. Consider an RLC Circuit with $R = 125\ \Omega$, $L = 1$ H, $C = 0.0004$ F, and a battery supplying $e_0 = 75$. Additionally, suppose $I(0) = 0, I'(0) = 75$. At time $t = 0$ the switch is closed and at time $t = 1$ it is opened and left open. This circuit can be modeled by

$$LI'' + RI' + \frac{1}{C}I = e'(t)$$

where $e'(t) = -75\delta(t-1)$.

(a) The delta function $\delta(t-a)$ satisfies

$$\int_0^\infty f(t)\delta(t-a)\ dt = f(a).$$

Using the definition of the Laplace transform, find $\mathscr{L}\{\delta(t-a)\}$.

(b) Find the resulting current $I(t)$ in the circuit using Laplace transform methods.