The Absorption of Gamma Rays by Matter

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Abstract

We investigate the probabilistic nature of radioactive decay, and study the interaction of lead with gamma rays produced by the decay of Cesium-137 (^{137}Cs). We found the frequencies of decays in a fixed time interval to match the predictions of Poisson counting statistics with a significance of 42%. By measuring the amount of gamma rays transmitted through varying thickness of lead, we find the intensity to decay exponentially with thickness. From a plot of this relationship, we find the coefficient of absorption for lead to be 1.088 ± 0.020 cm^{-1} at the relevant energy. Comparing with known values of the absorption coefficient we determine the energy of the incident gamma rays to be 0.7056 ± 0.0079 MeV.

1 Introduction

In about 95% of its decays, ^{137}Cs will undergo beta-decay into a metastable isotope of Barium (^{137m}Ba), which will in turn decay to ^{137}Ba and a 0.662 MeV gamma ray. In the other case, it will decay directly to ^{137}Ba releasing a β^- particle. This is summarized in Figure 1.

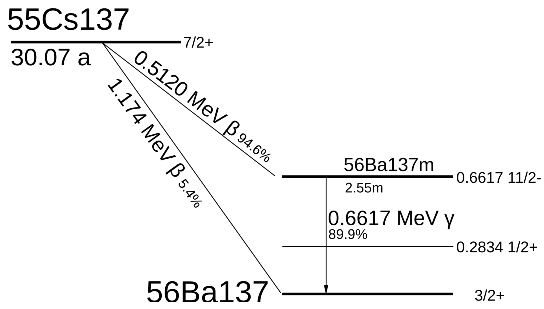

Figure 1: Decay products of ^{137}Cs

It would be useful to be able to predict when ^{137}Cs will decay, however it appears to decay by complete chance. If ^{137}Cs does decay spontaneously, we could describe the statistical nature with a Poisson distribution. Knowing the underlying statistics will give us great insight into the behavior of the decay, therefore it will be worthwhile to compare the predictions of Poisson statistics with experimental measurements.

1.1 Poisson Statistics and Goodness of Fit

If the decay is completely spontaneous, then for any fixed time interval Δt we would expect to see the same number of decays. However, for any random process you will get fluctuations about the expected value. The Poisson distribution described by:

$$P(m) = e^{-\lambda}\frac{\lambda^m}{m!}$$

is the probability of observing m events when the expectation is λ. The Poisson distribution has useful properties, namely the mean value and the variance of m is λ. To examine whether the decays follow Poisson statistics, we will count the number of decays in a fixed time interval and assign each repeated count a frequency. After N counts, we will have a probability distribution where the probability of each count is given by its frequency.

With a χ^2 test we can compare our measured probability density with the Poisson distribution. χ^2 is given by:

$$\frac{(n_1 - Np_1)^2}{Np_1} + \frac{(n_1 - Np_1)^2}{Np_1} + ... + \frac{(n_k - Np_k)^2}{Np_k} = \sum_{i=1}^{k} \frac{(n_i - Np_i)^2}{Np_i} \tag{1}$$

Where n_i is the measured frequency of the ith count, N is the total number of trials, p_i is the Poisson probability of seeing i counts, and k is the number of different counts or bins. If our hypothesis is true, that is the decay follows Poisson statistics, the probability of measuring the χ^2 value determined by equation 1 is given by a probability distribution $f(\chi^2)$. We can then define α as:

$$\int_{\chi^2_\alpha}^{\infty} f(\chi^2)d(\chi^2) = \alpha$$

That is, α is probability of measuring χ^2 greater than a particular χ^2_α. Therefore, we can obtain a χ^2 from equation 1 and find the corresponding α value to represent the probability of measuring a larger χ^2 in a repeated experiment. If our measured χ^2 corresponds to $\alpha = 0.50$ then there is a 50% chance of measuring a χ^2 larger than that, a reasonable result. An α of about 0.15 would be needed to confidently reject our hypothesis.

If the hypothesis is not rejected, the error in our counts can be estimated with the standard deviation of the Poisson distribution, that is $\sigma_{counts} = \sqrt{counts}$

1.2 Absorption of Gamma Rays by Lead

As the beam of 0.662 MeV gamma rays indicated in Figure 1 pass through a block of lead, they can either be absorbed, scattered away, or continue unaffected. We can place a Geiger-Muller Tube (GM-Tube) on the other side of the lead and count the number of gamma rays that passed unaffected. The change in the count, or intensity, through a slab of material is expected be proportional to the initial intensity I_0 and the thickness of the slab:

$$\Delta I = -\mu I_0 \Delta x \tag{2}$$

Where Δx is the thickness of the slab, and μ is a constant of proportionality. For a given Δx, larger μ corresponds to a larger change in intensity and μ has units of inverse length. The quantity μ can therefore be described as the absorption of gamma rays per unit length, and is known as the *absorption coefficient.* μ is generally dependent on the incident gamma ray energy.

There are three primary processes responsible for absorption of gamma rays in a material: Pair Production, Photoelectric Effect, and Compton Scattering. The energy required for pair production is at least the rest mass of an electron plus a positron or 1.022 MeV, far greater than the energy of the gamma rays.

The Photoelectric Effect is the absorption of gamma rays by electrons, causing the electrons to be kicked out of the material. The released electron (β^-) can scatter other gamma rays away from the GM-tube further decreasing the count. This interaction is known as Compton Scattering. Compton Scattering can also increase the count since the β^- particles themselves can be detected by the GM-tube. This contribution will be suppressed with increasing lead thickness, as β^- will interact with the lead atoms through Coulomb Scattering. But if the thickness is small enough, the photoelectric effect can produce β^- particles without them being scattered away. Therefore, we expect a slight increase of intensity with a small thickness barrier compared to no shielding at all.

To find μ we integrate equation 2:

$$I = I_0 e^{-\mu x} \tag{3}$$

Taking the natural logarithm, we obtain:

$$ln(I) = -\mu x + ln(I_0) \tag{4}$$

This is the equation of a straight line with whose slope is μ. Fitting a line to a plot of the logarithm of our intensity measurements versus thickness will enable us to extract μ.

Since the absorption coefficient is generally dependent on the gamma ray energy, we can compare our measured value of μ to a plot of photon energy versus standard values for the absorption coefficient for lead.

2 Apparatus

The setup of our experiment, shown in Figure 2, consists of a Geiger-Muller Tube, Tektronix Oscilloscope, a High Voltage Supply, and a Universal Counter.

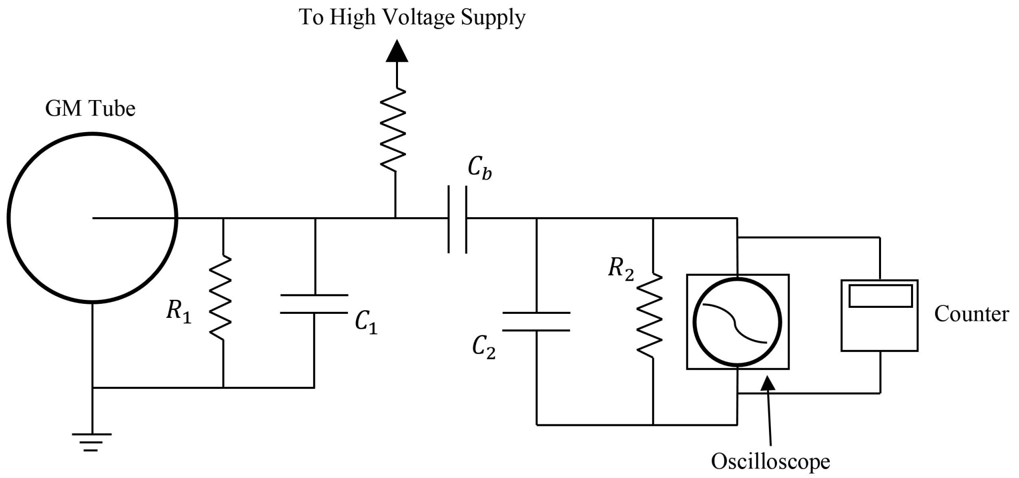

Figure 2: Circuit Diagram of Experiment. C_1 and C_2 are the internal capacitance of the GM tube and the oscilloscope. R_1 and R_2 are the internal resistances of the GM tube and oscilloscope. C_b is the blocking capacitor. Although not shown, the counter has internal resistance and capacitance as well.

2.1 Components

The GM tube is a conducting cylinder with a small central conducting wire placed concentric to cross-sections of the cylindrical surface. There is a high voltage difference between the surface and the wire, with the wire held at a positive potential, and the surface at a negative potential. The tube is filled with a rare gas that when struck by a gamma ray, will ionize. As they move toward the central wire, the free electrons will then ionize other atoms, each of which will produce more electrons. Because the positive ions are significantly heavier than the electrons, they will accumulate around the wire, shielding the electric field between the wire (anode) and the surface (cathode) and reducing the number of avalanches.

The amount of electrons collected on the anode only depends on the voltage difference, thus any ionizing radiation will produce the same amount of charge on the anode. The GM-Tube will take some time to reset the high voltage after the particles have discharged through it. This time interval is known as the *dead time*, denoted as τ, since no counts will be registered during its duration.

As the negatively charged electrons hit the anode, the internal capacitors charge up creating a negative voltage across the equivalent capacitor C. If the oscilloscope and counter are connected in parallel, the voltage across $C = C_1 + C_2 + C_{counter}$ will produce a signal measured by both components when discharged through their respective internal resistances.

The counter will trigger a count when a voltage of selected amplitude and slope is detected. As mentioned, the voltage signal and its slope are negative for each pulse. A trigger level of 6.0 corresponds to a minimum amplitude of 0.0 volts, higher levels correspond to negative amplitudes, and lower levels raises the threshold to positive amplitudes. The oscilloscope will be used to measure the discharge time of the capacitors, and the dead time of the GM-Tube.

3 Procedure

3.1 Preliminary Experiments

3.1.1 Observe the pulse shape and determine the number of electrons per pulse

If we arrange the electronics as shown in Figure 2 above but disconnect the universal counter, the oscilloscope will measure the voltage across $C = C_1 + C_2$. Each pulse will charge up the capacitors and after a time τ the voltage will rise back up to zero as the capacitors discharge through the equivalent resistance R given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. The time constant is given by $\tau = RC$ and can be measured using the oscilloscope. If we place the source near the tube, Figure 3 is the expected pulse displayed by the oscilloscope, along with the time constant. With R_1 and R_2 known, we can use the cursors of the oscilloscope to measure τ and obtain the capacitance C.

$$C_1 + C_2 = \frac{\tau}{R} = \frac{1.36ms}{0.7674M\Omega} = 0.177nF$$

To measure the number of electrons per pulse, we connect another capacitor C_k with a known capacitance (0.001μ F), such that $C_k \gg C_1 + C_2$. Then number of electrons is given by:

$$n = \frac{\Delta V C_k}{e} = \frac{0.436V \times 0.001\mu F}{1.60217 \times 10^{-19}C} = 2.72 \times 10^{-9}$$

Where e is the charge of the electron, $\Delta V = Q/C_k + C_1 + C_2$ and the amplitude of the pulse displayed on the oscilloscope. For the rest of the experiment, we disconnect C_k and in its place connect a 10 kΩ resistor.

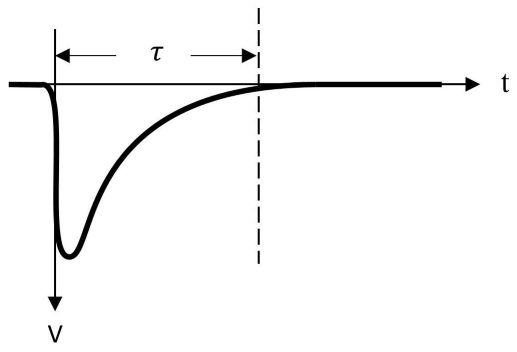

Figure 3: The pulse shape displayed by the oscilloscope. We measured the time constant as shown.

3.1.2 Measuring the counting rate

As mentioned in section 2, the pulse depends on the applied voltage but we wish to remove any fluctuations in count readings by the voltage. We reconnect the electronic counter in parallel with the oscilloscope and set the trigger level higher than 6.0 to remove noise pulses. Then we measure the count rate while varying the voltage to find a "plateau" region in which the count rate does not significantly change with neighboring voltage values. The center of this region was found to be approximately 880 volts. We also found the highest counting rate to be when the window of the tube was directly above the source. One side of the source gave higher counts than the other. This is due to a thin coating masking β^- particles on one side but not the other.

3.1.3 Investigating the counting statistics

To verify the hypothesis in section 1.1, we place a Cs^{137} source under the tube and move the tube such that at least 5 counts are detected per second. In 100 trials, we measure the number of decays in intervals of 1 second. With the 100 measurements, we can construct a frequency distribution with a histogram whose bins constitute the different measured counts. The result is displayed in Figure 5 in section 4.1. The height of each bin is the frequency of the count.

3.1.4 Measuring the counter dead time

The GM-tube will not detect any signals during the dead time τ, therefore when we detect n particles per second, the number of particles entering the tube will be:

$$m = \frac{n}{1 - n\tau}$$

Depending on the magnitude of τ, this can make our results misleading. We can measure the dead time directly using the "persist" mode on the oscilloscope. This mode will keep traces on the screen as shown in Figure 4. The dead time can be estimated as the time it takes for the pulse to recover to half its height. This is also indicated in Figure 4.

By employing two sources a and b and measuring their counts n_a,n_b, as well as their combined count n_{ab}, we can solve the above equation with the relation $m_{ab} = m_a + m_b$ to find an expression for the dead time.

$$\tau = \frac{1}{n_{ab}} \left\{ 1 - \left[1 - \frac{n_{ab}(n_a + n_b - n_{ab}}{n_a n_b}) \right]^{\frac{1}{2}} \right\} \tag{5}$$

When measuring the counts, we can place the sources next to each other instead of stacking them to prevent shielding and movement when swapping them out. Also, we place the sources such that the side shielding β^- particles is facing up. Placing the head of the tube as close as possible to the source(s) will also reduce the amount of scattering.

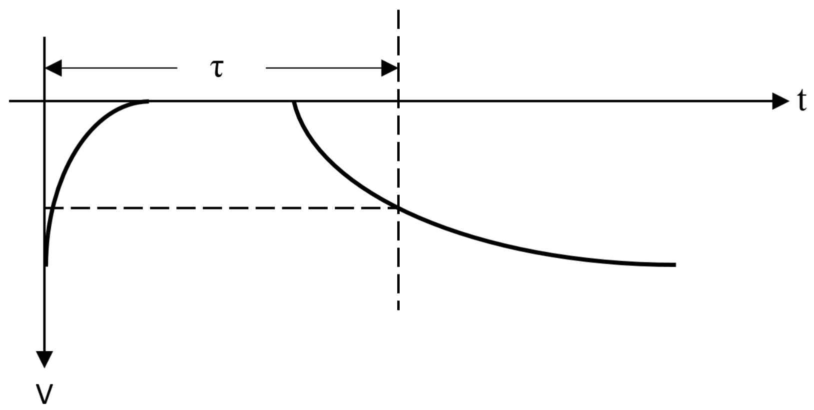

Figure 4. The dead time τ should be measured from half amplitude, depicted by the dashed horizontal line.

3.2 The absorption of gamma rays by lead

We begin by measuring the background radiation, then a trial with just the source, followed by a trial with a thin sheet of lead. The surface of the sheets are warped, thus measuring the thickness directly will be inaccurate. Instead the thickness can be better estimated by measuring the mass m with a scale, looking up the density ρ, and measuring surface area A with vernier calipers:

$$x = \frac{m}{\rho A} \tag{6}$$

It is critical that the source is placed such that the side that lets β^- particles through is facing away from the detector. As more sheets are added, the thickness should increase by at least 5 mm, with a maximum thickness of about 3 absorption lengths, or about 30 mm. However, the sheets of lead used were not thick enough, thus we stacked sheets of different thicknesses to generate the desired thickness. Since the air in between the sheets will not contribute to μ we only need to measure the thicknesses separately and add them. Nevertheless, the GM-tube should be positioned to allow for this extra height before adding in any sheets as the tube should not be moved. With our maximum thickness at 37 mm, we placed our tube roughly 42 mm above the base.

As we increase the thickness, we must also increase the amount of time spent counting to ensure we have enough counts. This is due to the error involved in counting statistics, that is $\sigma_{counts} = \sqrt{counts}$ and thus the relative error $\sqrt{counts}/counts = 1/\sqrt{counts}$. About 1000 counts at each thickness will yield a relative error of about 3%.

4 Data/Analysis

4.1 Fitting Decay Statistics to Poisson Statistics

Figure 5 displays the result of the measured frequency distribution along with $P(m)$. Evaluating equation 2 with $k = 17$ we obtain a χ^2:

$$\frac{\chi^2}{\nu} = 15.505/15 = 1.03369$$

Where $\nu = k - r$ are the number of degrees of freedom with r as the number of constraints. For fitting Poisson statistics, $r = 2$ since $\sigma_m^2 = \overline{m}$ and $\sum n_i = N$, thus $\nu = 15$ with 17 bins. Using an online α calculator [1] from χ^2 values, we obtain an $\alpha = 0.42$. Therefore, we have no reason to reject our hypothesis, as the corresponding confidence level is only 58%.

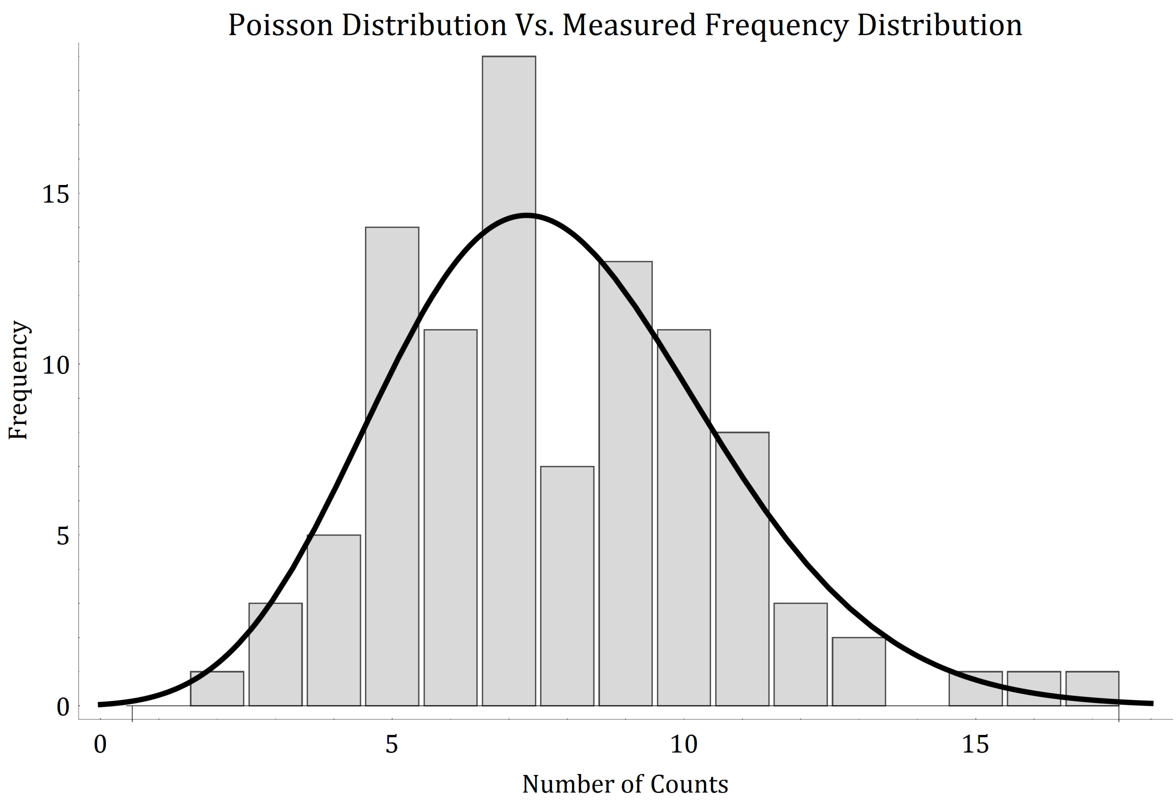

Figure 5. Poisson distribution plotted with a histogram of 100 trials of measuring counts in 1 sec intervals. There are a total of 17 bins.

4.2 Dead Time

Measuring with the oscilloscope gave us a dead time $\tau = 430$ μs for one source, 450 μs for the other, and 360 μs for both sources, with an average of $\overline{\tau} = 413$ μs.

n_a	n_b	n_{ab}
165	415	540
184	404	520
178	405	504
$\overline{n_a}$	$\overline{n_b}$	$\overline{n_{ab}}$
176	408	521

Table 1: Counts from two sources a and b, and then the two together. n_a is the count rate for source a, n_b is the count rate for source b, and n_{ab} is the count rate for both. The average of 3 trials are listed under $\overline{n_a}$,$\overline{n_b}$, and $\overline{n_{ab}}$.

The count rate for source a and b individually along with the count rate for both sources are listed in Table 1. With these values, we calculate a dead time of $\tau = 500.024$ using equation 5. To find the error in the dead time we use the general methods of error propagation, where the error of a function f with dependencies, $x_1, x_2, ..., x_N$, is given as:

$$\sigma_f^2 = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 \tag{7}$$

f for the dead time is given by equation 5. However, looking at Table 1, we find $n_{ab}(n_a + n_b - n_{ab}) < n_a n_b$, thus we can approximate equation 5 using binomial expansion:

$$\tau \approx \frac{n_a + n_b - n_{ab}}{2 n_a n_b}$$

Partial differentiating this function with n_a,n_b, and n_{ab} and making use of Poisson statistics to estimate the error in each, we obtain the error in the dead time:

$$\sigma_\tau \approx \frac{1}{2 n_a n_b} \sqrt{\frac{(n_{ab} - n_b)^2}{n_a} + \frac{(n_{ab} - n_a)^2}{n_b} + n_{ab}}$$

Using the average values found in Table 1 and the above equation, we obtain an error of $\sigma_\tau =$ 200 μs. The large error is due to each term under the square root being larger than 100, with the last term being 521. The square root ($\sim$ 10) is divided by $n_a n_b$ ($\sim 10^5$), resulting in an error on the order of 10^{-4} s or 100 μs. Applying the error propagation equation (7) directly to the full definition of τ gives an error of 280 μs. The calculation of this was performed in Mathematica and is included in the Appendix.

Consequently, our expected dead time is $\tau =$ 500 $\pm$ 300 μs. Our averaged measured value of 413 μs falls within this range.

4.3 Thickness of Lead

From weighing each slice, measuring the surface area, and using the known density for lead, we can calculate the thickness of each slice more precisely than measuring directly. The results are listed in Table 2.

Mass (g)	Length(cm)	Width(cm)	Thickness (cm)
58.9	7.747	7.747	0.0865
783.5	10.2362	10.3378	0.653
1495	10.414	10.3632	1.22
777.6	10.16	10.3378	0.653
776.5	10.2616	10.2616	0.650
1489.4	10.2362	10.2616	1.25

Table 2: Measured values of mass, length, and width of each slice along with their calculated thickness.

The mass was measuring using a scale with a precision of 0.1 g. The length and width of each slice was measured with a vernier caliper up to 0.001 in, or 0.00254 cm. To find the error in thickness, we use equation 7 with f defined by equation 6 where $A = \text{length}(l) \times \text{width}(w)$:

$$\sigma_x = \frac{1}{\rho l w}\sqrt{\sigma_m^2 + \frac{m^2}{w^2}\sigma_w^2 + \frac{m^2}{l^2}\sigma_l^2}$$

Where $\rho = 11.34\ g/cm^3$. The errors for m, l, and w were estimated by the precision of the instrument used, thus $\sigma_m = 0.1$ g and $\sigma_l = \sigma_w = 0.003$ cm. The with these approximations, the relative error in the thickness x is under 3%.

4.4 Intensity vs Thickness

In a 500 s interval, we measured the background count to be 180 counts. The background intensity, or count rate, is then $180/500s = 0.36s^{-1}$. By stacking slabs of lead each with a thickness labeled in Table 2, we were able to get a shield of lead with thicknesses at regular intervals up to 38 mm. The thickness along with corresponding intensities are displayed in Table 3. The Intensity here is defined as:

$$I = \frac{c}{t} - \frac{c_b}{t_b} \tag{8}$$

Where c is the number of counts measured in a time t, and c_b is the number of background counts in the time t_b. From equation 4, the natural log of intensity should be plotted with x and a line of best fit weighted by the relevant errors will produce μ. As stated in the previous section, the relative error in x is less than 3%, therefore we can assume x to be an exact quantity. The line will then only be weighted by the error in $ln(I_i)$.

x (cm)	Counts	Time (s)	Intensity with background	Intensity I	$ln(I)$	σ_{lnI}
0.0865	928	100	9.28	8.92	2.19	0.034
0.653	995	200	4.98	4.62	1.53	0.035
1.222	1106	400	2.77	2.41	0.88	0.036
1.875	926	600	1.54	1.18	0.17	0.048
2.528	983	1000	0.98	0.62	-0.47	0.066
3.178	951	1380	0.69	0.33	-1.11	0.106
3.778	984	1800	0.55	0.19	-1.68	0.171

Table 3: Our measured counts for each thickness x along with the calculated intensities. The background intensity was $0.36 \pm 0.03 s^{-1}$.

The calculated values for $ln(I_i)$ and its error are also shown in Table 3. The error was found by applying equation 7 to the natural logarithm of equation 8:

$$\sigma_{ln(I)} = \sqrt{\left(\frac{c}{t} - \frac{b}{t_b}\right)^{-2} \left(\frac{c}{t^2} + \frac{b}{t_b^2}\right)}$$

As χ^2 was used to fit decay statistics to the Poisson distribution, it can also be used to fit the data in Table 3 to a line $ln(I) = ax + b$. The χ^2 in this case is similar to equation 1:

$$\chi^2 = \sum_{i=1}^{7} \frac{ln(I_i) - (ax_i + b)}{\sigma_{ln(I_i)}}$$

The constants a and b are to be determined and will correspond to the absorption coefficient and the no-lead intensity, as shown in equation 4 in the introduction. We can find a and b by minimizing χ^2, that is taking the partial derivative of χ^2 with respect to each variable and setting it equal to zero. From [2], this leaves us with two equations in a and b:

$$a = \frac{1}{\Delta}(W \cdot P - X_1 \cdot Y_1) \qquad b = \frac{1}{\Delta}(X_2 \cdot Y_1 - X_1 \cdot P) \tag{9}$$

To simplify the result, we have represented sums with the following:

$$W \equiv \sum w_i \qquad X_1 \equiv \sum w_i x_i \qquad Y_1 \equiv \sum w_i y_i$$
$$X_2 \equiv \sum w_i x_i^2 \qquad Y_2 \equiv \sum w_i y_i^2 \qquad P \equiv \sum w_i x_i y_i$$
$$\Delta \equiv W \cdot X_2 - X_1^2$$

Where $w_i = 1/\sigma_i^2$, with σ_i being shorthand for the error in $ln(I_i)$. To estimate the error in a and b , we propagate the error in $ln(I_i)$ using equation 7 with the definitions of a and b, given by equation 9.

$$\sigma_a^2 = \frac{W}{\Delta} \qquad \sigma_b^2 = \frac{X_2}{\Delta} \tag{10}$$

With equations 9 and 10, we find $\mu = 1.088 \pm 0.020$ cm^{-1} and $I_0 = 2.247 \pm 0.028$ we find our fit to minimize the χ^2 to a value 5.204. Fitting a line involves two constraints a and b thus the degrees of freedom are $k - 2$. With $k = 7$ data points, we find $\chi^2/\nu = 1.0408$, leading to an $\alpha = 0.39$. This gives only a 61% confidence level of rejecting our data, therefore we must compare our result to the true value.

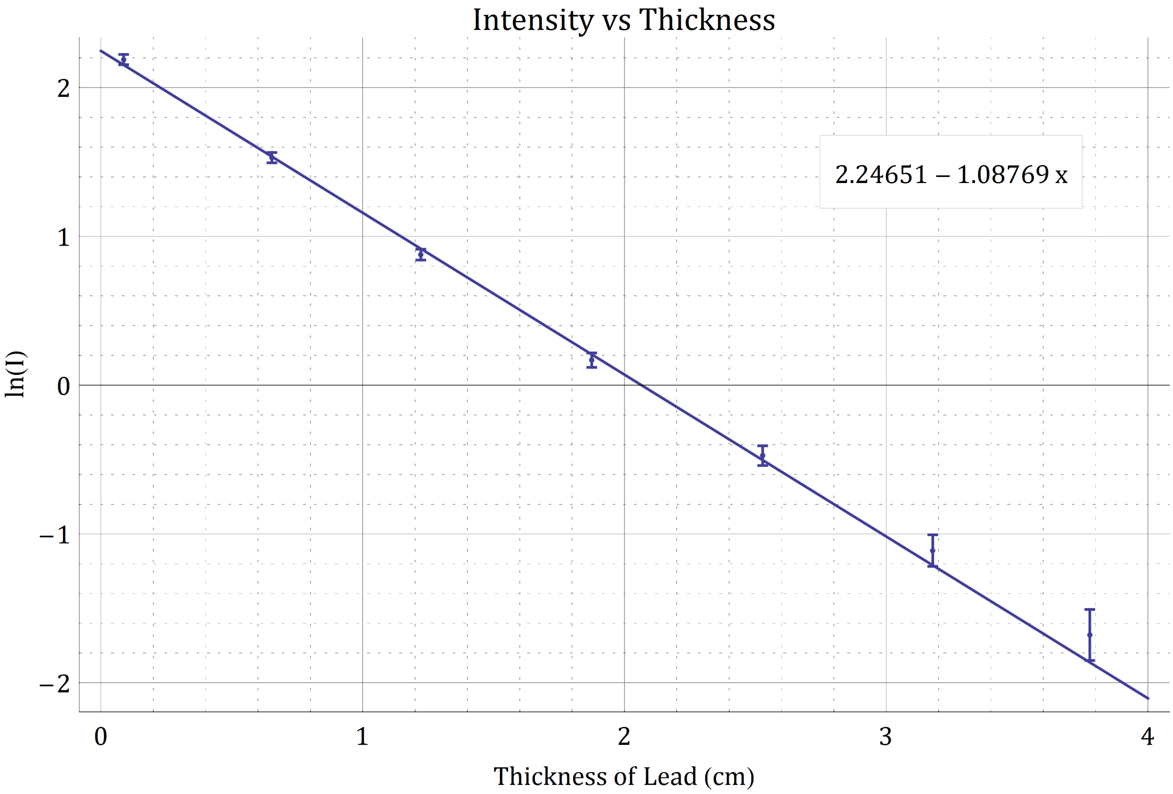

Figure 6. Logarithm of measured intensity plotted against absorber thickness. The slope μ is 1.088 ± 0.020 cm^{-1}.

Photon Energy (MeV)	μ (cm^{-1})	Photon Energy (MeV)	μ (cm^{-1})
1.022	59.9	1.022	0.771
0.1277	33.6	1.362	0.62
0.1703	16.4	2.043	0.499
0.2554	6.31	4.086	0.469
0.3405	3.39	5.108	0.491
0.4086	2.42	10.22	0.607
0.5108	1.68	15.32	0.698
0.6811	1.16	25.54	0.835

Table 4: Values of Lead's Absorption Coefficient compared with their corresponding photon energies.

As mentioned in the introduction, μ is generally dependent on the gamma ray energy. Table 4 lists the coefficient of absorption of lead for different photon energies. Figure 7 is a plot of a range of this list, along with a 2^{nd} order interpolation. The plot, ranging from 0 to 1.2 MeV, is used to obtain an energy of 0.7056 ± 0.010 MeV for our measured μ. The true value, 0.662 MeV, is about 4.4 standard deviations away.

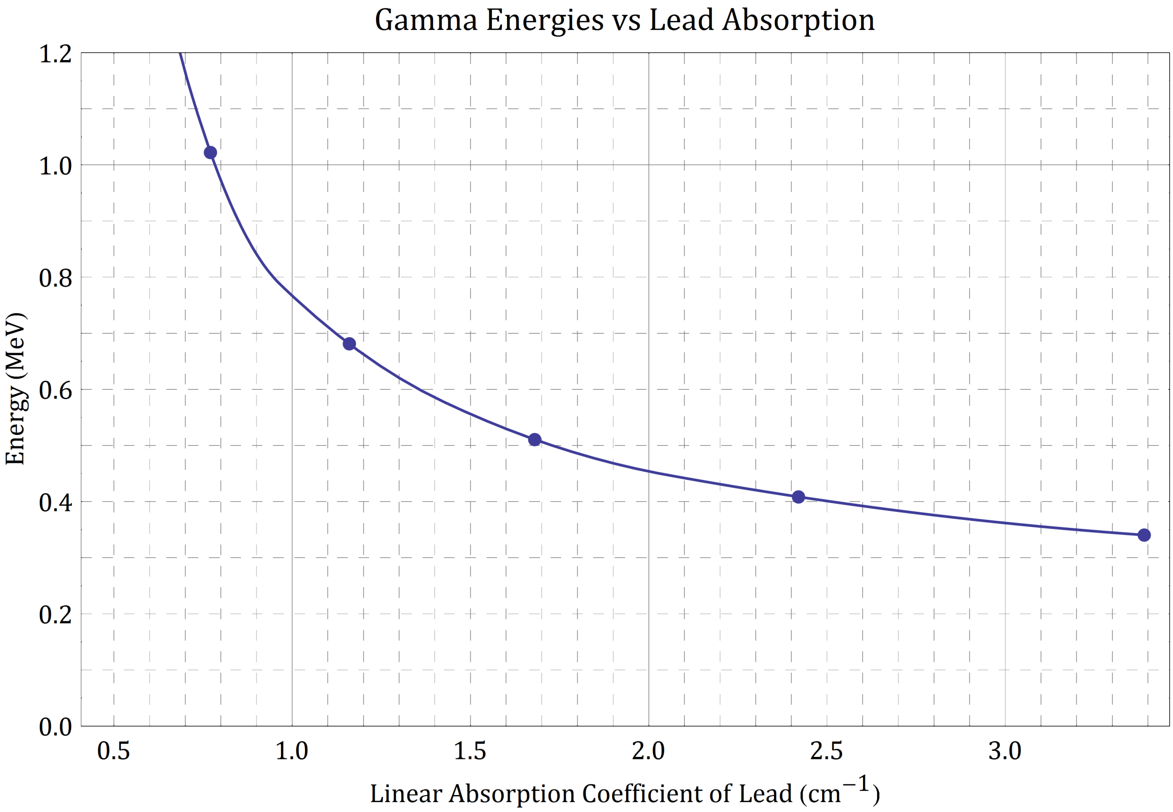

Figure 7. Gamma Ray Energies for Values of the Absorption Coefficient

5 Conclusion

Here, we were able to determine the rate at which lead absorbs gamma rays per cm with a Geiger-Muller Tube. In the setup of our experiment, we found setting 880 volts across the tube to produce reliable results. By measuring the decay time of the capacitors to be 1.36ms, we establish 2.72×10^{-9} electrons charge up the tube with each pulse. We found the dead time of the tube to be $500 \pm 300\mu$s. To evaluate the error involved in counting gamma ray intensity, we modeled Cs^{137} decay with Poisson statistics and found a significance of 42% in our hypothesis. Finally, we found lead's coefficient of absorption μ to be 1.088 ± 0.02 cm^{-1}. By looking up standard values, we deduce the energy of the incident gamma rays to be 0.7056 ± 0.010 MeV, roughly 7% from the accepted value.

A Appendix

A.1 Dead Time Error

The following the calculation run in Mathematica to calculate the dead time error. ka corresponds the counts from source a, or $ka = n_a$. Similarly $ki = n_i$ with $i = (a, b, ab)$.

```
(Debug) In[12]:=
        ClearAll[ka, kb, kab];
        τ1[ka_, kb_, kab_] := 1/kab (1 - Sqrt[1 - kab/(ka kb) (ka + kb - kab)]);
        (* Definition of Dead Time *)

        da1 = Simplify[D[τ1[ka, kb, kab], ka]];
        (* Partial Derivative of τ with respect to counts of source a *)
        db1 = Simplify[D[τ1[ka, kb, kab], kb]];
        (* Partial Derivative of τ with respect to counts of source b *)
        dab1 = Simplify[D[τ1[ka, kb, kab], kab]];
        (* Partial Derivative of τ with respect to counts of both sources a,b *)

        ka = 176;
        kb = 408;
        kab = 521;
        (* Values for na, nb, nab *)

        Errorrrr1 = Sqrt[da1^2 ka + db1^2 kb + dab1^2 kab] * 10^6;
        (*calculates the error according to equation 7 and converts to microseconds *)

        N[Errorrrr1] (* Outputs the error in dead time *)
        dddeedtime1 = N[τ1[ka, kb, kab]] * 10^6 (* Displays the dead time*)

(Debug) Out[21]=
        278.153

(Debug) Out[22]=
        505.141
```

References

[1] Soper, D.S. (2015). p-Value Calculator for a Chi-Square Test [Software]. Available from http://www.danielsoper.com/statcalc

[2] Fred Kuttner and George Brown. Physics 133, 2014.