UNIT – II Fundamentals of Logic: 7 Hours

- Basic Connectives and Truth Tables,
- Logic Equivalence
- The Laws of Logic,
- Logical Implication
- Rules of Inference

UNIT II 7 Hours

Fundamentals of Logic

Introduction:

Propositions:

A proposition is a declarative sentence that is either true or false (but not both). For instance, the following are propositions: "Paris is in France" (true), "London is in Denmark" (false), "2 < 4" (true), "4 = 7 (false)". However the following are not propositions: "what is your name?" (this is a question), "do your homework" (this is a command), "this sentence is false" (neither true nor false), "x is an even number" (it depends on what x represents), "Socrates" (it is not even a sentence). The truth or falsehood of a proposition is called its truth value.

Basic Connectives and Truth Tables:

Connectives are used for making compound propositions. The main ones are the following (p and q represent given propositions):

Name	Represented	Meaning
Negation	$\neg p$	"not p"
Conjunction	$p \wedge q$	"p and q"
Disjunction	$p \vee q$	"p or q (or both)"
Exclusive Or	$p \oplus q$	"either p or q, but not both"
Implication	$p \longrightarrow q$	"if p then q"
Biconditional	$p \leftrightarrow q$	"p if and only if q"

The truth value of a compound proposition depends only on the value of its components. Writing F for "false" and T for "true", we can summarize the meaning of the connectives in the following way:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow p$	$\leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Note that $\vee$ represents a non-exclusive or, i.e., $p \vee q$ is true when any of p, q is true and also when both are true. On the other hand ⊕ represents an exclusive or, i.e., p ⊕ q is true only when exactly one of p and q is true.

Tautology, Contradiction, Contingency:

1. A proposition is said to be a tautology if its truth value is T for any assignment of truth values to its components. Example : The proposition p ∨ ¬p is a tautology.

2. A proposition is said to be a contradiction if its truth value is F for any assignment of truth values to its components. Example : The proposition p ∧ ¬p is a contradiction.

3. A proposition that is neither a tautology nor a contradiction is called a contingency.

p	¬p	p ∨ ¬p	p ∧ ¬p
T	F	T	F
T	F	T	F
F	T	T	F
F	T	T	F

tautology contradiction

Conditional Propositions: A proposition of the form "if p then q" or "p implies q", represented "p → q" is called a conditional proposition. For instance: "if John is from Chicago then John is from Illinois". The proposition p is called hypothesis or antecedent, and the proposition q is the conclusion or consequent.

Note that p → q is true always except when p is true and q is false. So, the following sentences are true: "if 2 < 4 then Paris is in France" (true → true), "if London is in Denmark then 2 < 4" (false → true),

"if 4 = 7 then London is in Denmark" (false → false). However the following one is false: “if $2 < 4$ then London is in Denmark” (true → false).

In might seem strange that “p → q” is considered true when p is false, regardless of the truth value of q. This will become clearer when we study predicates such as “if x is a multiple of 4 then x is a multiple of 2”. That implication is obviously true, although for the particular
case $x = 3$ it becomes “if 3 is a multiple of 4 then 3 is a multiple of 2”.

The proposition p ↔ q, read “p if and only if q”, is called bicon- ditional. It is true precisely when p and q have the same truth value, i.e., they are both true or both false.

Logical Equivalence: Note that the compound proposi- tions $p \to q$ and $\neg p \vee q$ have the same truth values:

p	q	¬p	¬p ∨ q	p →
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

When two compound propositions have the same truth values no matter what truth value their constituent propositions have, they are called logically equivalent. For instance $p \to q$ and $\neg p \vee q$ are logically equivalent, and we write it:

$$p \to q \equiv \neg p \vee q$$

Note that that two propositions A and B are logically equivalent precisely when $A \leftrightarrow B$ is a tautology.

Example : De Morgan’s Laws for Logic. The following propositions are logically equivalent:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	¬p	¬q	p ∨ q	¬(p ∨ q)	¬p ∧ ¬q	p ∧ q	¬(p ∧ q)	¬p ∨ ¬q
T	T	F	F	T	F	F	T	F	F
T	F	F	T	T	F	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	T	T	F	T	T	F	T	T

Example : The following propositions are logically equivalent:

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Again, this can be checked with the truth tables:

p	q	p → q	q → p	(p → q) ∧ (q → p)	p ↔ q
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Exercise : Check the following logical equivalences:

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$
$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$
$$\neg(p \leftrightarrow q) \equiv p \oplus q$$

Converse, Contrapositive: The converse of a conditional proposition $p \rightarrow q$ is the proposition $q \rightarrow p$. As we have seen, the bi- conditional proposition is equivalent to the conjunction of a conditional proposition an its converse.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

So, for instance, saying that "John is married if and only if he has a spouse" is the same as saying "if John is married then he has a spouse" and "if he has a spouse then he is married".

Note that the converse is not equivalent to the given conditional proposition, for instance “if John is from Chicago then John is from Illinois” is true, but the converse “if John is from Illinois then John is from Chicago” may be false.
The contrapositive of a conditional proposition $p \rightarrow q$ is the propo- sition $\neg q \rightarrow \neg p$. They are logically equivalent. For instance the con- trapositive of “if John is from Chicago then John is from Illinois” is “if
John is not from Illinois then John is not from Chicago”.
LOGICAL CONNECTIVES: New propositions are obtained with the aid of word or phrases like “not”,”and”,”if….then”,and “if and only if”. Such words or phrases are called logical connectives. The new propositions obtained by the use of connectives are called compound propositions. The original propositions from which a compound proposition is obtained are called the components or the primitives of the compound proposition. Propositions which do not contain any logical connective are called simple propositions

NEGATION: A Proposition obtained by inserting the word “not” at an appropriate place in a given proposition is called the negation of the given proposition. The negation of a proposition p is denoted by ~p(read “not p”)

Ex: p: 3 is a prime number

~p: 3 is not a prime number

Truth Table:

p	~p
0	1
1	0

CONJUNCTION:
A compound proposition obtained by combining two given propositions by inserting the word “and” in between them is called the conjunction of the given proposition.The conjunction of two proposition p and q is denoted by p^q(read “p and q”).

- The conjunction p^q is true only when p is true and q is true; in all other cases it is false.
- Ex: p:√2 is an irational number q: 9 is a prime number
  p^q: √2 is an irational number and 9 is a prime number
- Truth table:

p	q	p^q
0	0	0
0	1	0

1	0	0
1	1	1

DISJUNCTION:

A compound proposition obtained by combining two given propositions by inserting the word "or" in between them is called the disjunction of the given proposition.The disjunction of two proposition p and q is denoted by $p \vee q$(read "p or q").

- The disjunction $p \vee q$ is false only when p is false and q is false ; in all other cases it is true.

- Ex: p:√2 is an irational number q: 9 is a prime number
  $p \vee q$: √2 is an irational number or 9 is a prime number Truth table:

-

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

EXCLUSIVE DISJUNCTION:

- The compound proposition "p or q" to be true only when either p is true or q is true but not both. The exclusive or is denoted by symbol $\underline{v}$.

- Ex: p:√2 is an irrational number q: 2+3=5

$P\underline{v}q$: Either √2 is an irrational number or 2+3=5 but not both.

- Truth Table:

p	q	$p\underline{v}q$
0	0	0
0	1	1
1	0	1
1	1	0

CONDITIONAL(or IMPLICATION):

- A compound proposition obtained by combining two given propositions by using the words "if" and "then" at appropriate places is called a conditional or an implication..

- Given two propositions p and q, we can form the conditionals “if p, then q” and “if q, then p:. The conditional “if p, then q”is denoted by p→q and the conditional “if q, then p” is denoted by q →p.
- The conditional p→q is false only when p is true and q is false ;in all other cases it is true.
- Ex: p: 2 is a prime number q: 3 is a prime number

p→q: If 2 is a prime number then 3 is a prime number; it is true

- Truth Table:

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

BICONDITIONAL:

- Let p and q be two propositions,then the conjunction of the conditionals p→q and q→p is called bi-conditional of p and q. It is denoted by p↔q.
- p↔q is same as (p→q)∧ (q→p). As such p↔q is read as “ if p then q and if q then p”.
- Ex: p: 2 is a prime number q: 3 is a prime number p↔q are true.

Truth Table:

p	q	p→q	q→p	p↔q
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

COMBINED TRUTH TABLE

P	q	~p	p^q	p∨q	p⊻q	p→q	p↔q
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0

1 1 0 1 1 0 1 1

TAUTOLOGIES; CONTRADICTIONS:

A compound proposition which is always true regardless of the truth values of its components is called a tautology.

A compound proposition which is always false regardless of the truth values of its components is called a contradiction or an absurdity.

A compound proposition that can be true or false (depending upon the truth values of its components) is called a contingency I.e contingency is a compound proposition which is neither a tautology nor a contradiction.

LOGICAL EQUIVALENCE

- Two propositions 'u' and 'v' are said to be logically equivalent whenever u and v have the same truth value, or equivalently .
- Then we write u□ v. Here the symbol □ stands for "logically equivalent to".
- When the propositions u and v are not logically equivalent we write u□ v.

LAWS OF LOGIC:

To denote a tautology and To denotes a contradiction.

- Law of Double negation: For any proposition p,(~~p)□ p
- Idempotent laws: For any propositions p, 1) (p□ p) □ p 2) (p□ p) □ p
- Identity laws: For any proposition p, 1)(p□ Fo) □ p 2)(p□ To) □ p
- Inverse laws: For any proposition p, 1) (p □ □ p) □ To 2)(p□ ~p)□ Fo
- Commutative Laws: For any proposition p and q, 1)(p□ q)□ (q□ p) 2)(p□ q)□ (q□ p)
- Domination Laws: For any proposition p, 1) (p□ To) □ To 2) (p□ Fo) □ Fo
- Absorption Laws: For any proposition p and q,1) [p□ (p□ q)] □ p 2)[p□ (p□ q)] □ p
- De-Morgan Laws: For any proposition p and q, 1)~ (p□ q)□ □p□ □q 2) □(p□ q)□ □p □ □q

- Associative Laws : For any proposition p ,q and r, 1) p □ (q □ r) □ (p □ q) □ r 2) p□ (q□ r)□ (p□ q) □ r

- Distributive Laws: For any proposition p ,q and r, 1) p □ (q □ r) □ (p □ q) □ (p □ r) 2) p□ (q□ r)□ (p□ q) □ (p□ r)

- Law for the negation of a conditional : Given a conditional p→q, its negation is obtained by using the following law: □(p→q)□ [p□ (□q)]

NOTE:

- ~ (p□ q)≡ □p□ □q

- □(p□ q) ≡ □p □ □ q

- □(p→q) ≡ [p□ (□q)]

- (p→q) ≡ □ □ (p→q) ≡ □[p□ (□q)] ≡□p □ q

TRANSITIVE AND SUBSTITUTION RULES If u,v,w are propositions such that u□ v and v □ w, then u □ w. (this is transitive rule)

- Suppose that a compound proposition u is a tautology and p is a component of u, we replace each occurrence of p in u by a proposition q, then the resulting compound proposition v is also a tautology(This is called a substitution rule).

- Suppose that u is a compound proposition which contains a proposition p. Let q be a proposition such that q □ p , suppose we replace one or more occurrences of p by q and obtain a compound proposition v. Then u □ v (This is also substitution rule)

APPLICATION TO SWITCHING NETWORKS

- If a switch p is open, we assign the symbol o to it and if p is closed we assign the symbol 1 to it.

- Ex: current flows from the terminal A to the terminal B if the switch is closed i.e if p is assigned the symbol 1. This network is represented by the symbol p

A P B

- Ex: parallel network consists of 2 switches p and q in which the current flows from the terminal A to the terminal B , if p or q or both are closed i.e if p or q (or both) are assigned the symbol 1. This network is represent by p□ q

Ex: Series network consists of 2 switches p and q in which the current flows from the terminal A to the terminal B if both of p and q are closed; that is if both p and q are assigned the symbol 1. This network is represented by p□ q

DUALITY:

Suppose u is a compound proposition that contains the connectives □ and □ . Suppose we replace each occurrence of □ and □ in u by □ and □ respectively.

Also if u contains To and Fo as components, suppose we replace each occurrence of To and Fo by Fo and To respectively, then the resulting compound proposition is called the dual of u and is denoted by u^d.

Ex: u: p □ (q □ □r) □ (s □ To) u^d: p □ (q □ □ r) □ (s □ Fo)

NOTE:

- $(u^d)^d$ □ u. The dual of the dual of u is logically equivalent to u.
- For any two propositions u and v if u □ v, then u^d □ v^d . This is known as the principle of duality.

The connectives NAND and NOR

(p ↑q)= □(p □ q) □ □ p □ □ q

(p ↓q)= □(p □ q) □ □ p □ □ q

CONVERSE,INVERSE AND CONTRAPOSITIVE

Consider a conditional (p→q) , Then :

1) q→p is called the converse of p→q

2) □p→□q is called the inverse of p→q

3) □q→□p is called the contrapositive of p→q

RULES OF INFERENCE:

There exist rules of logic which can be employed for establishing the validity of arguments . These rules are called the Rules of Inference.

1) Rule of conjunctive simplification: This rule states that for any two propositions p and q if $p\wedge q$ is true, then p is true i.e $(p\wedge q)\Rightarrow p$.

2) Rule of Disjunctive amplification: This rule states that for any two proposition p and q if p is true then $p\vee q$ is true i.e $p\Rightarrow (p\vee q)$

3) 3) Rule of Syllogism: This rule states that for any three propositions p,q r if $p\to q$ is true and $q\to r$ is true then $p\to r$ is true. i.e $\{(p\to q)\wedge (q\to)\}\Rightarrow (p\to r)$ In tabular form:

$p\to q$ $q\to r$ $\therefore (p\to r)$

4) 4) Modus pones(Rule of Detachment): This rule states that if p is true and $p\to q$ is true, then q is true, ie $\{p\wedge (p\to q)\}\Rightarrow q$. Tabular form

p $p\to q$ $\therefore q$

5) Modus Tollens: This rule states that if $p\to q$ is true and q is false, then p is false. $\{(p\to q)\wedge \neg q\}\Rightarrow q$ Tabular form: $p\to q$

$\neg q$ $\therefore \neg p$

6) Rule of Disjunctive Syllogism: This rule states that if $p\vee q$ is true and p is false, then q is true i.e. $\{(p\vee q)\wedge \neg p\}\Rightarrow q$ Tabular Form $p\vee q$

$\neg p$ $\therefore q$

QUANTIFIERS:

1. The words "ALL","EVERY","SOME","THERE EXISTS" are called quantifiers in the proposition

2. The symbol $\forall$ is used to denote the phrases "FOR ALL","FOR EVERY","FOR EACH" and "FOR ANY".this is called as universal quantifier.

3. $\exists$ is used to denote the phrases "FOR SOME"and "THERE EXISTS"and "FOR ATLEAST ONE".this symbol is called existential quantifier.

A proposition involving the universal or the existential quantifier is called a quantified statement

LOGICAL EQUIVALENCE:

1. □ x,[p(x)□ q(x)]□ (□ x p(x))□ (□ x,q(x))

2. □ x, [p(x)□ q(x)]□ (□ x p(x)) □ (□ x,q(x))

3. □ x, [p(x)→q(x)] □ □ x ,[□p(x)□ q(x)]

RULE FOR NEGATION OF A QUANTIFIED STATEMENT:

□{□ x,p(x)}≡□ x{□p(x)} □{□ x,p(x)}≡□ x{□p(x)}

RULES OF INTERFERENCE:

1. RULE OF UNIVERSAL SPECIFICATION

2. RULE OF UNIVERSAL GENERALIZATION

If an open statement p(x) is proved to be true for any (arbitrary)x chosen from a set S, then the quantified statement □ x€S, p(x) is true.

METHODS OF PROOF AND DISPROOF:

1.DIRECT PROOF:

The direct method of proving a conditional p→q has the following lines of argument:

a) hypothesis : First assume that p is true

b) Analysis: starting with the hypothesis and employing the rules /laws of logic and other known facts , infer that q is true

c) Conclusion:p→q is true.

2. INDIRECT PROOF:

Condition p→q and its contrapositive □q→□p are logically equivalent. On basis of this proof, we infer that the conditional p→q is true. This method of proving a conditional is called an indirect method of proof.

3.PROOF BY CONTRADICTION

The indirect method of proof is equivalent to what is known as the proof by contradiction. The lines of argument in this method of proof of the statement p→q are as follows:

1) Hypothesis: Assume that p→q is false i.e assume that p is true and q is false.

2)Analysis: starting with the hypothesis that q is false and employing the rules of logic and other known facts , infer that p is false. This contradicts the assumption that p is true

3)Conculsion: because of the contradiction arrived in the analysis , we infer that p→q is true

4.PROOF BY EXHAUSTION:

"∀ x *€S*,p(x)" is true if p(x)is true for every (each) x in S.If S consists of only a limited number of elements , we can prove that the statement "∀ x *€S*,p(x)" is true by considering p(a) for each a in S and verifying that p(a) is true .such a method of prove is called method of exhaustion.

5.PROOF OF EXISTENCE:

"∃ x *€S*,p(x)" is true if any one element a € S such that p(a) is true is exhibited. Hence , the best way of proving a proposition of the form "∃ x *€S*,p(x)" is to exhibit the existence of one a€S such that p(a) is true. This method of proof is called proof of existence.

6.DISPROOF BY CONTRADICTION :

Suppose we wish to disprove a conditional p→q. for this propose we start with the hypothesis that p is true and q is true, and end up with a contradiction. In view of the contradiction , we conclude that the conditional p→q is false.this method of disproving p→q is called DISPROOF BY CONTRADICTION

7.DISPROOF BY COUNTER EXAMPLE:

"∀ x €S,p(x)" is false if any one element a€S such that p(a) is false is exhibited hence the best way of disproving a proposition involving the universal quantifiers is to exhibit just one case where the proposition is false. This method of disproof is called DISPROOF BY COUNTER EXAMPLE