Exercise 4. Let $D \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary and consider the heat equation with Neumann boundary conditions:

$$\begin{cases} u_t - \Delta u = f(\mathbf{x}) & \text{in } D \quad (1a) \\ \nabla u \cdot \hat{n} = 0 & \text{on } \partial D \quad (1b) \end{cases}$$

From the divergence theorem and (1b) we find that

$$\begin{aligned} \int_D \nabla \cdot (\nabla u) \, \mathrm{d}\mathbf{x} &= \int_{\partial D} (\nabla u \cdot \hat{n}) \, \mathrm{d}S(\mathbf{x}) \\ &= \int_{\partial D} 0 \cdot \mathrm{d}S(\mathbf{x}) \\ &= 0 \,. \end{aligned}$$

We can rewrite (1a) as $\Delta u = u_t - f(\mathbf{x})$ and, noting that the Laplacian $\Delta \equiv \nabla \cdot \nabla$, take volume integrals over D on all terms such that

$$\begin{aligned} \int_D \nabla \cdot (\nabla u) \, \mathrm{d}\mathbf{x} &= \int_D u_t \, \mathrm{d}\mathbf{x} - \int_D f(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ 0 &= \int_D u_t \, \mathrm{d}\mathbf{x} - \int_D f(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ \int_D f(\mathbf{x}) \, \mathrm{d}\mathbf{x} &= \int_D u_t \, \mathrm{d}\mathbf{x} \,. \end{aligned}$$

If $f(\mathbf{x})$ is strictly positive (i.e. always a source, never a sink), then we find with the boundary conditions (1b) that

$$\int_D f(\mathbf{x}) \, \mathrm{d}\mathbf{x} > 0 \Rightarrow \int_D u_t \, \mathrm{d}\mathbf{x} > 0 \,.$$

Thus the solution to (1), $u = u(t, \mathbf{x})$, is strictly growing over time in D. From this we can claim that

$$\lim_{t \to \infty} \int_D u(t, \mathbf{x}) \, \mathrm{d}\mathbf{x} = +\infty \,.$$