

EXAMPLE 1 All the following declarative sentences are propositions.



1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false. ◀

Some sentences that are not propositions are given in Example 2.

EXAMPLE 2 Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

DEFINITION 1 Let p be a proposition. The *negation of p* , denoted by $\neg p$ (also denoted by \bar{p}), is the statement
 “It is not the case that p .”

The proposition $\neg p$ is read “not p .” The truth value of the negation of p , $\neg p$, is the opposite of the truth value of p .

EXAMPLE 3 Find the negation of the proposition

“Today is Friday.”



and express this in simple English.

Solution: The negation is

“It is not the case that today is Friday.”

This negation can be more simply expressed by

“Today is not Friday,”

or

“It is not Friday today.”

EXAMPLE 4 Find the negation of the proposition

“At least 10 inches of rain fell today in Miami.”

and express this in simple English.

Solution: The negation is

“It is not the case that at least 10 inches of rain fell today in Miami.”

This negation can be more simply expressed by

Truth Table for the Negation of a Proposition.	
p	$\neg p$
T	F
F	T

DEFINITION 2 Let p and q be propositions. The *conjunction* of p and q , denoted by $p \wedge q$, is the proposition “ p and q .” The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

EXAMPLE 5 Find the conjunction of the propositions p and q where p is the proposition “Today is Friday” and q is the proposition “It is raining today.”

Solution: The conjunction of these propositions, $p \wedge q$, is the proposition “Today is Friday and it is raining today.” This proposition is true on rainy Fridays and is false on any day that is not a Friday and on Fridays when it does not rain. ◀

DEFINITION 3 Let p and q be propositions. The *disjunction* of p and q , denoted by $p \vee q$, is the proposition “ p or q .” The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 The Truth Table for the Disjunction of Two Propositions.		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE 6 What is the disjunction of the propositions p and q where p and q are the same propositions as in Example 5?



Solution: The disjunction of p and q , $p \vee q$, is the proposition

“Today is Friday or it is raining today.”

This proposition is true on any day that is either a Friday or a rainy day (including rainy Fridays). It is only false on days that are not Fridays when it also does not rain. ◀

As was previously remarked, the use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, in an inclusive way. Thus, a disjunction is true when at least one of the two propositions in it is true. Sometimes, we use *or* in an exclusive sense. When the exclusive or is used to connect the propositions p and q , the proposition “ p or q (but not both)” is obtained. This proposition is true when p is true and q is false, and when p is false and q is true. It is false when both p and q are false and when both are true.

DEFINITION 4 Let p and q be propositions. The *exclusive or* of p and q , denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.		
p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.		
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional Statements

We will discuss several other important ways in which propositions can be combined.

DEFINITION 5 Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition “if p , then q .” The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.

The truth table for the conditional statement $p \rightarrow q$ is shown in Table 5. Note that the statement $p \rightarrow q$ is true when both p and q are true and when p is false (no matter what truth value q has).

Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \rightarrow q$. You will encounter most if not all of the following ways to express this conditional statement:

“if p , then q ”	“ p implies q ”
“if p , q ”	“ p only if q ”
“ p is sufficient for q ”	“a sufficient condition for q is p ”
“ q if p ”	“ q whenever p ”
“ q when p ”	“ q is necessary for p ”
“a necessary condition for p is q ”	“ q follows from p ”
“ q unless $\neg p$ ”	

A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is

“If I am elected, then I will lower taxes.”

Similarly, consider a statement that a professor might make:

“If you get 100% on the final, then you will get an A.”

EXAMPLE 7 Let p be the statement “Maria learns discrete mathematics” and q the statement “Maria will find a good job.” Express the statement $p \rightarrow q$ as a statement in English.



Solution: From the definition of conditional statements, we see that when p is the statement “Maria learns discrete mathematics” and q is the statement “Maria will find a good job,” $p \rightarrow q$ represents the statement

“If Maria learns discrete mathematics, then she will find a good job.”

There are many other ways to express this conditional statement in English. Among the most natural of these are:

“Maria will find a good job when she learns discrete mathematics.”

“For Maria to get a good job, it is sufficient for her to learn discrete mathematics.”

and

“Maria will find a good job unless she does not learn discrete mathematics.” ◀

“If it is sunny today, then we will go to the beach.”

are statements used in normal language where there is a relationship between the hypothesis and the conclusion. Further, the first of these statements is true unless Maria learns discrete mathematics, but she does not get a good job, and the second is true unless it is indeed sunny today, but we do not go to the beach. On the other hand, the statement

“If today is Friday, then $2 + 3 = 5$.”

is true from the definition of a conditional statement, because its conclusion is true. (The truth value of the hypothesis does not matter then.) The conditional statement

“If today is Friday, then $2 + 3 = 6$.”

is true every day except Friday, even though $2 + 3 = 6$ is false.

We would not use these last two conditional statements in natural language (except perhaps in sarcasm), because there is no relationship between the hypothesis and the conclusion in either

statement. In mathematical reasoning, we consider conditional statements of a more general sort than we use in English. The mathematical concept of a conditional statement is independent of a cause-and-effect relationship between hypothesis and conclusion. Our definition of a conditional statement specifies its truth values; it is not based on English usage. Propositional language is an artificial language; we only parallel English usage to make it easy to use and remember.

The if-then construction used in many programming languages is different from that used in logic. Most programming languages contain statements such as **if p then S** , where p is a proposition and S is a program segment (one or more statements to be executed). When execution of a program encounters such a statement, S is executed if p is true, but S is not executed if p is false, as illustrated in Example 8.

EXAMPLE 8 What is the value of the variable x after the statement

if $2 + 2 = 4$ then $x := x + 1$

if $x = 0$ before this statement is encountered? (The symbol $:=$ stands for assignment. The statement $x := x + 1$ means the assignment of the value of $x + 1$ to x .)

Solution: Because $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value $0 + 1 = 1$ after this statement is encountered. ◀

CONVERSE, CONTRAPOSITIVE, AND INVERSE We can form some new conditional statements starting with a conditional statement $p \rightarrow q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$. The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$. The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$. We will see that of these three conditional statements formed from $p \rightarrow q$, only the contrapositive always has the same truth value as $p \rightarrow q$.

EXAMPLE 9 What are the contrapositive, the converse, and the inverse of the conditional statement

“The home team wins whenever it is raining.”?

Solution: Because “ q whenever p ” is one of the ways to express the conditional statement $p \rightarrow q$, the original statement can be rewritten as

“If it is raining, then the home team wins.”

Consequently, the contrapositive of this conditional statement is

“If the home team does not win, then it is not raining.”

The converse is

“If the home team wins, then it is raining.”

The inverse is

“If it is not raining, then the home team does not win.”

Only the contrapositive is equivalent to the original statement. ◀

BICONDITIONALS We now introduce another way to combine propositions that expresses that two propositions have the same truth value.

DEFINITION 6 Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition “ p if and only if q .” The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

The truth table for $p \leftrightarrow q$ is shown in Table 6. Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise. That is why we use the words “if and only if” to express this logical connective and why it is symbolically written by combining the symbols \rightarrow and \leftarrow . There are some other common ways to express $p \leftrightarrow q$:

“ p is necessary and sufficient for q ”
 “if p then q , and conversely”
 “ p iff q .”

The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation “iff” for “if and only if.” Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

EXAMPLE 10 Let p be the statement “You can take the flight” and let q be the statement “You buy a ticket.” Then $p \leftrightarrow q$ is the statement

“You can take the flight if and only if you buy a ticket.”



This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket and cannot take the flight (such as when the airline bumps you). ◀

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Tables of Compound Propositions

EXAMPLE 11 Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.					
p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

TABLE 8 Precedence of Logical Operators.	
<i>Operator</i>	<i>Precedence</i>
\neg	1
\wedge	2
\vee	3
\rightarrow	4
\leftrightarrow	5

Translating English Sentences

EXAMPLE 12 How can this English sentence be translated into a logical expression?

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

Solution: There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as p , this would not be useful when analyzing its meaning or reasoning with it. Instead, we will use propositional variables to represent each sentence part and determine the appropriate logical connectives between them. In particular, we let a , c , and f represent “You can access the Internet from campus,” “You are a computer science major,” and “You are a freshman,” respectively. Noting that “only if” is one way a conditional statement can be expressed, this sentence can be represented as

$$a \rightarrow (c \vee \neg f).$$



EXAMPLE 13 How can this English sentence be translated into a logical expression?

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”

Solution: Let q , r , and s represent “You can ride the roller coaster,” “You are under 4 feet tall,” and “You are older than 16 years old,” respectively. Then the sentence can be translated to

$$(r \wedge \neg s) \rightarrow \neg q.$$

Of course, there are other ways to represent the original sentence as a logical expression, but the one we have used should meet our needs. ◀

System Specifications

Translating sentences in natural language (such as English) into logical expressions is an essential part of specifying both hardware and software systems. System and software engineers take requirements in natural language and produce precise and unambiguous specifications that can be used as the basis for system development. Example 14 shows how compound propositions can be used in this process.

EXAMPLE 14 Express the specification “The automated reply cannot be sent when the file system is full” using logical connectives.



Solution: One way to translate this is to let p denote “The automated reply can be sent” and q denote “The file system is full.” Then $\neg p$ represents “It is not the case that the automated reply can be sent,” which can also be expressed as “The automated reply cannot be sent.” Consequently, our specification can be represented by the conditional statement $q \rightarrow \neg p$. ◀

System specifications should be **consistent**, that is, they should not contain conflicting requirements that could be used to derive a contradiction. When specifications are not consistent, there would be no way to develop a system that satisfies all specifications.

EXAMPLE 15 Determine whether these system specifications are consistent:

- “The diagnostic message is stored in the buffer or it is retransmitted.”
- “The diagnostic message is not stored in the buffer.”
- “If the diagnostic message is stored in the buffer, then it is retransmitted.”

Solution: To determine whether these specifications are consistent, we first express them using logical expressions. Let p denote “The diagnostic message is stored in the buffer” and let q denote “The diagnostic message is retransmitted.” The specifications can then be written as $p \vee q$, $\neg p$, and $p \rightarrow q$. An assignment of truth values that makes all three specifications true must have p false to make $\neg p$ true. Because we want $p \vee q$ to be true but p must be false, q must be true. Because $p \rightarrow q$ is true when p is false and q is true, we conclude that these specifications are consistent because they are all true when p is false and q is true. We could come to the same conclusion by use of a truth table to examine the four possible assignments of truth values to p and q . ◀

EXAMPLE 16 Do the system specifications in Example 15 remain consistent if the specification “The diagnostic message is not retransmitted” is added?

Solution: By the reasoning in Example 15, the three specifications from that example are true only in the case when p is false and q is true. However, this new specification is $\neg q$, which is false when q is true. Consequently, these four specifications are inconsistent. ◀

Boolean Searches



Logical connectives are used extensively in searches of large collections of information, such as indexes of Web pages. Because these searches employ techniques from propositional logic, they are called **Boolean searches**.

In Boolean searches, the connective *AND* is used to match records that contain both of two search terms, the connective *OR* is used to match one or both of two search terms, and the connective *NOT* (sometimes written as *AND NOT*) is used to exclude a particular search term. Careful planning of how logical connectives are used is often required when Boolean searches are used to locate information of potential interest. Example 17 illustrates how Boolean searches are carried out.

EXAMPLE 17 Web Page Searching Most Web search engines support Boolean searching techniques, which usually can help find Web pages about particular subjects. For instance, using Boolean searching to find Web pages about universities in New Mexico, we can look for pages matching *NEW AND MEXICO AND UNIVERSITIES*. The results of this search will include those pages that contain the three words *NEW*, *MEXICO*, and *UNIVERSITIES*. This will include all of the pages of interest, together with others such as a page about new universities in Mexico. (Note that in Google, and many other search engines, the word “AND” is not needed, although it is understood, because all search terms are included by default.) Next, to find pages that deal with universities in New Mexico or Arizona, we can search for pages matching *(NEW AND MEXICO OR ARIZONA) AND UNIVERSITIES*. (Note: Here the *AND* operator takes precedence over the *OR* operator. Also, in Google, the terms used for this search would be *NEW MEXICO OR ARIZONA*.) The results of this search will include all pages that contain the word *UNIVERSITIES* and either both the words *NEW* and *MEXICO* or the word *ARIZONA*. Again, pages besides those of interest will be listed. Finally, to find Web pages that deal with universities in Mexico (and not New Mexico), we might first look for pages matching *MEXICO AND UNIVERSITIES*, but because the results of this search will include pages about universities in New Mexico, as well as universities in Mexico, it might be better to search for pages matching *(MEXICO AND UNIVERSITIES) NOT NEW*. The results of this search include pages that contain both the words *MEXICO* and *UNIVERSITIES* but do not contain the word *NEW*. (In Google, and many other search engines, the word “NOT” is replaced by a minus sign “-”. In Google, the terms used for this last search would be *MEXICO UNIVERSITIES - NEW*.) ◀



Logic Puzzles



Puzzles that can be solved using logical reasoning are known as **logic puzzles**. Solving logic puzzles is an excellent way to practice working with the rules of logic. Also, computer programs designed to carry out logical reasoning often use well-known logic puzzles to illustrate their capabilities. Many people enjoy solving logic puzzles, which are published in books and periodicals as a recreational activity.

We will discuss two logic puzzles here. We begin with a puzzle that was originally posed by Raymond Smullyan, a master of logic puzzles, who has published more than a dozen books containing challenging puzzles that involve logical reasoning.

EXAMPLE 18 In [Sm78] Smullyan posed many puzzles about an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people *A* and *B*. What are *A* and *B* if *A* says “*B* is a knight” and *B* says “The two of us are opposite types”?



Solution: Let p and q be the statements that A is a knight and B is a knight, respectively, so that $\neg p$ and $\neg q$ are the statements that A is a knave and that B is a knave, respectively.

We first consider the possibility that A is a knight; this is the statement that p is true. If A is a knight, then he is telling the truth when he says that B is a knight, so that q is true, and A and B are the same type. However, if B is a knight, then B 's statement that A and B are of opposite types, the statement $(p \wedge \neg q) \vee (\neg p \wedge q)$, would have to be true, which it is not, because A and B are both knights. Consequently, we can conclude that A is not a knight, that is, that p is false.

If A is a knave, then because everything a knave says is false, A 's statement that B is a knight, that is, that q is true, is a lie, which means that q is false and B is also a knave. Furthermore, if B is a knave, then B 's statement that A and B are opposite types is a lie, which is consistent with both A and B being knaves. We can conclude that both A and B are knaves. ◀

We pose more of Smullyan's puzzles about knights and knaves in Exercises 55–59 at the end of this section. Next, we pose a puzzle known as the **muddy children puzzle** for the case of two children.

EXAMPLE 19 A father tells his two children, a boy and a girl, to play in their backyard without getting dirty. However, while playing, both children get mud on their foreheads. When the children stop playing, the father says "At least one of you has a muddy forehead," and then asks the children to answer "Yes" or "No" to the question: "Do you know whether you have a muddy forehead?" The father asks this question twice. What will the children answer each time this question is asked, assuming that a child can see whether his or her sibling has a muddy forehead, but cannot see his or her own forehead? Assume that both children are honest and that the children answer each question simultaneously.

Solution: Let s be the statement that the son has a muddy forehead and let d be the statement that the daughter has a muddy forehead. When the father says that at least one of the two children has a muddy forehead, he is stating that the disjunction $s \vee d$ is true. Both children will answer "No" the first time the question is asked because each sees mud on the other child's forehead. That is, the son knows that d is true, but does not know whether s is true, and the daughter knows that s is true, but does not know whether d is true.

After the son has answered "No" to the first question, the daughter can determine that d must be true. This follows because when the first question is asked, the son knows that $s \vee d$ is

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

true, but cannot determine whether s is true. Using this information, the daughter can conclude that d must be true, for if d were false, the son could have reasoned that because $s \vee d$ is true, then s must be true, and he would have answered “Yes” to the first question. The son can reason in a similar way to determine that s must be true. It follows that both children answer “Yes” the second time the question is asked. ◀

Logic and Bit Operations

<i>Truth Value</i>	<i>Bit</i>
T	1
F	0

Computers represent information using bits. A **bit** is a symbol with two possible values, namely, 0 (zero) and 1 (one). This meaning of the word bit comes from *binary digit*, because zeros and ones are the digits used in binary representations of numbers. The well-known statistician John Tukey introduced this terminology in 1946. A bit can be used to represent a truth value, because there are two truth values, namely, *true* and *false*. As is customarily done, we will use a 1 bit to represent true and a 0 bit to represent false. That is, 1 represents T (true), 0 represents F (false). A variable is called a **Boolean variable** if its value is either true or false. Consequently, a Boolean variable can be represented using a bit.

Links



Computer **bit operations** correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators \wedge , \vee , and \oplus , the tables shown in Table 9 for the corresponding bit operations are obtained. We will also use the notation *OR*, *AND*, and *XOR* for the operators \vee , \wedge , and \oplus , as is done in various programming languages.

Information is often represented using bit strings, which are lists of zeros and ones. When this is done, operations on the bit strings can be used to manipulate this information.

DEFINITION 7 A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

EXAMPLE 20 101010011 is a bit string of length nine. ◀

We can extend bit operations to bit strings. We define the **bitwise OR**, **bitwise AND**, and **bitwise XOR** of two strings of the same length to be the strings that have as their bits the *OR*, *AND*, and *XOR* of the corresponding bits in the two strings, respectively. We use the symbols \vee , \wedge , and \oplus to represent the bitwise *OR*, bitwise *AND*, and bitwise *XOR* operations, respectively. We illustrate bitwise operations on bit strings with Example 21.

EXAMPLE 21 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 0110110110 and 1100011101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise *OR*, bitwise *AND*, and bitwise *XOR* of these strings are obtained by taking the *OR*, *AND*, and *XOR* of the corresponding bits, respectively. This gives us

01 1011 0110	
11 0001 1101	
<hr/>	
11 1011 1111	bitwise <i>OR</i>
01 0001 0100	bitwise <i>AND</i>
10 1010 1011	bitwise <i>XOR</i>



1.2 Propositional Equivalences

Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term “compound proposition” to refer to an expression formed from propositional variables using logical operators, such as $p \wedge q$.

We begin our discussion with a classification of compound propositions according to their possible truth values.

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Tautologies and contradictions are often important in mathematical reasoning. Example 1 illustrates these types of compound propositions.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

EXAMPLE 1 We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction. ◀

Logical Equivalences



Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. We can also define this notion as follows.

DEFINITION 2 The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol \equiv is not a logical connective and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol \leftrightarrow is sometimes used instead of \equiv to denote logical equivalence.

One way to determine whether two compound propositions are equivalent is to use a truth table. In particular, the compound propositions p and q are equivalent if and only if the columns giving their truth values agree. Example 2 illustrates this method to establish an extremely important and useful logical equivalence, namely, that $\neg(p \vee q)$ of $\neg p \wedge \neg q$. This logical equivalence is one of the two **De Morgan laws**, shown in Table 2, named after the English mathematician Augustus De Morgan, of the mid-nineteenth century.



EXAMPLE 2 Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution: The truth tables for these compound propositions are displayed in Table 3. Because the truth values of the compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q , it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these compound propositions are logically equivalent. ◀

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE 6 Logical Equivalences.	
<i>Equivalence</i>	<i>Name</i>
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

and

$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n).$$

(Methods for proving these identities will be given in Section 4.1.)

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditionals.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Using De Morgan's Laws

The two logical equivalences known as De Morgan's laws are particularly important. They tell us how to negate conjunctions and how to negate disjunctions. In particular, the equivalence $\neg(p \vee q) \equiv \neg p \wedge \neg q$ tells us that the negation of a disjunction is formed by taking the conjunction of the negations of the component propositions. Similarly, the equivalence $\neg(p \wedge q) \equiv \neg p \vee \neg q$ tells us that the negation of a conjunction is formed by taking the disjunction of the negations of the component propositions. Example 5 illustrates the use of De Morgan's laws.

EXAMPLE 5 Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution: Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then "Miguel has a cellphone and he has a laptop computer" can be represented by $p \wedge q$. By the first of De Morgan's laws, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. Consequently, we can express

the negation of our original statement as “Miguel does not have a cellphone or he does not have a laptop computer.”

Let r be “Heather will go to the concert” and s be “Steve will go to the concert.” Then “Heather will go to the concert or Steve will go to the concert” can be represented by $r \vee s$. By the second of De Morgan’s laws, $\neg(r \vee s)$ is equivalent to $\neg r \wedge \neg s$. Consequently, we can express the negation of our original statement as “Heather will not go to the concert and Steve will not go to the concert.” ◀

Constructing New Logical Equivalences

The logical equivalences in Table 6, as well as any others that have been established (such as those shown in Tables 7 and 8), can be used to construct additional logical equivalences. The reason for this is that a proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition. This technique is illustrated in Examples 6–8, where we also use the fact that if p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent (see Exercise 56).

EXAMPLE 6 Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.



Solution: We could use a truth table to show that these compound propositions are equivalent (similar to what we did in Example 4). Indeed, it would not be hard to do so. However, we want to illustrate how to use logical identities that we already know to establish new logical identities, something that is of practical importance for establishing equivalences of compound propositions with a large number of variables. So, we will establish this equivalence by developing a series of logical equivalences, using one of the equivalences in Table 6 at a time, starting with $\neg(p \rightarrow q)$ and ending with $p \wedge \neg q$. We have the following equivalences.

$$\begin{aligned} \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by Example 3} \\ &\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan law} \\ &\equiv p \wedge \neg q && \text{by the double negation law} \end{aligned}$$

EXAMPLE 7 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. (Note: we could also easily establish this equivalence using a truth table.) We have the following equivalences.

$$\begin{aligned} \neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\ &\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\ &\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\ &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\ &\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\ &\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\ &\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F} \end{aligned}$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent. ◀

EXAMPLE 8 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (*Note:* This could also be done using a truth table.)

$$\begin{aligned}(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\ &\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative} \\ &&& \text{laws for disjunction} \\ &\equiv \mathbf{T} \vee \mathbf{T} && \text{by Example 1 and the commutative} \\ &&& \text{law for disjunction} \\ &\equiv \mathbf{T} && \text{by the domination law} \quad \blacktriangleleft\end{aligned}$$