The Foundation

Predicates and Quantifiers

Introduction

- In mathematics, statements involving variables are extensively used.
  - Example: “ x>3”, “x=y-3”, etc.
  - in the statement “x>3” the variable x is called the subject and “ >3” or” greater than 3” is called the predicate of the subject.
  - “ x>3” can be represented by a propositional function p(x), also called predicate.
  - p(x) is true for every x such that x>3.
  - “x=y-3” could be represented by a propositional function q(x,y).
  - P(x,y) is true for every couple (x,y) such that x=y-3.

- A general form of a predicate is an n-tuple propositional function p(x1, x2, ..xn).

Universal Quantifiers

- The universal Quantifier of p(x)
  - is the proposition " p(x) is true for all values of x in the universe of discourse".
  - It is also expressed by
    " for all x p(x)" or " for every x p(x)"
  - It is denoted $\forall p(x)$
  - When the elements of the universe of discourse can be listed {x1, x2, …xn}, the quantification $\forall p(x)$ is the same as $p(x1) \wedge p(x2) \wedge \ldots p(xn)$.

The Existential Quantifier

- The existential Quantifier of p(x)
  - is the proposition " There exists an element x in the universe of discourse such that p(x) is true".
  - It is also expressed by
    " There is an x such as p(x)" or " There is at least one x such that p(x)" or " for some p(x).
  - It is denoted $\exists p(x)$
  - When the elements of the universe of discourse can be listed {x1, x2, …xn}, the quantification $\exists p(x)$ is the same as $p(x1) \vee p(x2) \vee \ldots p(xn)$.

Truth Values of Quantifications

Statement	**When true**	**When false**
$\forall x\ p(x)$	p(x) is true for every x	There is an x for which p(x) is false.
$\exists x\ p(x)$	There is an x for which p(x) is true	p(x) is false for every x.

Examples

- All S are P : $\forall x\ S(x) \rightarrow P(x)$
  - “All lions are fierce”
  - $\forall x\ l(x) \rightarrow f(x)$

- Some S are P : $\exists x\ (S(x) \wedge P(x))$
  - “Some lions do not drink coffee”
  - $\exists x\ (l(x) \wedge d(x))$

Examples

- " Every student in this class has studied Calculus"
  - If universe of discourse is students of this class -only-
    - C(x) denotes " x has studied calculus"
    - Proposition becomes $\forall x\, C(x)$
  - If universe of discourse is all student of the university
    - C(x) denotes " x has studied calculus"
    - CSS(x) denotes " x is a Computer Science student"
    - the proposition becomes: $\forall x\, CSS(x) \rightarrow C(x)$
- Translate the statement: $\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$
  - If universe of discourse for x and y is the set of all students in your school.
  - C(x): is " x has a computer"
  - F(x,y): is " x and y are friends"
  - The translation:" for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends"
  - I.e. " every student in your school has a computer or has a friend who has a computer"

Examples

- Translate the statements
  - $\exists x \forall y \forall z ((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)))$
    - F(x,y): " x and y are friends"
    - Universe of discourse (UD) is the set of all students
  - "There is a student none of whose friends are also friends"
  - " Everyone has exactly one best friend"
    - B(x,y) :"y is the best friend of x"
  - $\forall x \ \exists y \forall z (B(x,y) \wedge ((z \neq y) \rightarrow \neg B(x,z))$
  - "there is a women who has taken a flight on every airline in the world"
    - take(x,y):" x has taken y"
    - isaflight(x,y): " x is a flight on y"
  - $\exists x \forall y \exists z (take(x,z) \wedge isaflight(z,y))$

Combining Quantifiers

- A variable that is not assigned a value and no quantifier is applied on it is said to be **free** otherwise it is said to be **bound**.

- Assume p(x,y) is " x + y = 2"
  - $\forall x \, \forall y$ p(x,y) means "for all real numbers x and real numbers y it is true that "x + y = 2" " False
  - $\exists x \, \forall y$ p(x,y) means "there is a real number x such that for every real number y, " x+y=2" is true" False
  - $\forall x \, \exists y$ p(x,y) means "For every real number x, there is a number y such that p(x,y) is true" True
  - $\exists x \, \exists y$ p(x,y) means " there is a real number x and a real number y, such that "x+y=2" is true " True

Negations

- $\neg (\forall x\ P(x) \Leftrightarrow \exists x\ \neg P(x)$

- $\neg (\exists x\ P(x)) \Leftrightarrow \forall x\ \neg P(x)$