4 Limits

We want to understand how functions of real variable might behave under *small changes* of the variable. Given $f : \mathbb{R} \to \mathbb{R}$ we can ask if as the variable x gets closer and closer to a fixed value c, does the corresponding function values $f(x)$ also get closer and closer to some value? We can also consider if $f(x)$ approaches some value as we take larger and larger values of x. In this chapter we shall try to recognise *limiting* behaviour of a function and develop techniques for calculating limits.

Let's begin by looking at some examples.

Example 4.1. Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

As x approaches 0 from the right i.e. through positive values, $f(x)$ will always take the value . . If x approaches 0 from the left i.e. through negative values, $f(x)$ will always take the value . .

Example 4.2. Consider the function $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$. This is not defined when $x = 1$. Nonetheless, we can consider what happens to the function for values of x very close to 1.

x	1.01000	1.00100	1.00010	1.00001	0.9000	0.9900	0.9990	0.9999
$f(x)$								

This seems to suggest that $f(x)$ approaches . . . as x approaches 1 from the right, or from the left.

Note that $x^3 - 1 =$ . . . . and $x^2 - 1 =$ . . . . Thus if $x \neq 1$ then we have $f(x) =$ . . . ., which allows us to see why $f(x)$ approaches . . . as x approaches 1.

Algebraically,

$$\begin{aligned}\frac{x^3-1}{x^2-1} &= \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} \\ &= \frac{x^2+x+1}{x+1}, \quad \text{provided } x \neq 1.\end{aligned}$$

This approaches 3/2 as x approaches 1.

Alternatively, we can write a number close to 1 as $1+h$ where h is a non-zero small number. Then

$$\begin{aligned}\frac{x^3-1}{x^2-1} &= \frac{(1+h)^3-1}{(1+h)^2-1} \\ &= \frac{1+3h+3h^2+h^3-1}{1+2h+h^2-1} \\ &= \end{aligned}$$

Example 4.3. Consider the function $f(x) = |x - \frac{\pi}{2}|\tan x$. Then $f(\frac{\pi}{2})$ is not defined. But we can consider what happens for values of x close to $\frac{\pi}{2}$. Writing $x = \frac{\pi}{2} + h$, we need to consider $f(x)$ as h becomes small.

h	0.1	0.01	0.001	0.0001
$f(\frac{\pi}{2}+h)$	$-.9966644423$	$-.9999666664$	$-.9999996667$	$-.9999996667\dots$
$f(\frac{\pi}{2}-h)$	0.9966644423	0.9999666664	0.9999996667	$0.9999996667\dots$

The table above suggests that $f(x)$ approaches -1 as x approaches $\frac{\pi}{2}$ from the right i.e. values larger than $\frac{\pi}{2}$. However, when x approaches $\frac{\pi}{2}$ from the left, then $f(x)$ approaches 1.

Informal definition of limit. We write $\lim_{x \to c} f(x) = \ell$ and say f has limit ℓ as x approaches/tends to c if $f(x)$ approaches the value ℓ as x approaches c.

One sided limits are obtained by restricting how x approaches c. If $f(x)$ approaches the value ℓ as x approaches c from the right i.e. through values of $x > c$, then we say that the right hand limit of f at c is ℓ and write $\lim_{x \to c^+} f(x) = \ell$. Similarly for the left hand limit: $\lim_{x \to c^-} f(x) = \ell$ means $f(x)$ approaches the value ℓ as x approaches c from the left i.e. through values of $x < c$.

Remark. $x \to c$ is shorthand x approaches c. So $\lim_{x \to c} f(x) = \ell$ is shorthand for $f(x) \to \ell$ as $x \to c$.

Can we make this more precise? Numerically, we can choose a sequence $x_1, x_2, \ldots$, all different from c, approximating c. We then want the sequence $f(x_n)$ to approach ℓ. This should be independent of which sequence $x_1, x_2, \ldots$ we choose to approximate c. If you are happy with the idea of a sequence (x_n) converging to a limit, then: $\lim_{x \to c} f(x) = \ell$ means for every sequence $x_n \to c$ with $x_n \neq c$ we have $f(x_n) \to \ell$.

Similarly, $\lim_{x \to c^+} f(x) = \ell$ (respectively $\lim_{x \to c^-} f(x) = \ell$) means the sequence $f(x_n)$ gets closer and closer to ℓ whenever (x_n) is a sequence approaching c from the right (respectively from the left) i.e. $x_n > c$ (respectively $x_n < c$) for all n.

The following observation is specially useful when we have a function defined by different formulae on either side of c.

Proposition 4.4. $\lim_{x\to c} f(x) = \ell$ *if and only if* $\lim_{x\to c^+} f(x) = \ell = \lim_{x\to c^-} f(x)$.

Example 4.5.

(i) For the function $f : \mathbb{R} \to \mathbb{R}$ considered in Example 4.1, we have $\lim_{x\to 0^+} f(x) =$ . . . and $\lim_{x\to 0^-} f(x) =$ . . . So $\lim_{x\to 0} f(x)$ . . . .

(ii) For the function $f(x) = \dfrac{x^3-1}{x^2-1}$ from Example 4.2, we have $\lim_{x\to 0^+} f(x) =$ . . . and $\lim_{x\to 0^-} f(x) =$ . . . So $\lim_{x\to 0} f(x)$ . . . . . . What is a good choice to take for defining $f(1)$?

(iii) For the function $f(x) = (x - \frac{\pi}{2})\tan x$ from Example 4.3, we have $\lim_{x\to \frac{\pi}{2}^+} f(x) =$ . . . and $\lim_{x\to \frac{\pi}{2}^-} f(x) =$ . . . So $\lim_{x\to \frac{\pi}{2}} f(x)$ . . . . . .

It is essential that we remember the calculation of $\lim_{x\to c} f(x)$ does not require f to have a value at $x = c$. Even if $f(c)$ is defined, we completely ignore it. Furthermore, there is no reason why the limit—if it exists—should be the function value (even when the function value is defined). `BUT` it is very nice when they match, and we give it a name.

Definition 4.6. The function f is continuous at c if $\lim_{x\to c} f(x) = f(c)$.

We say that the function f is continuous on a subset S of the domain of f if f is continuous at every point in S.

Remark. Geometrically, continuity at a point sort of means that the graph doesn't have a break at that point. Most of the functions we meet will be continuous, as can be seen from their graphs.

Example 4.7. Consider the following function from Example 4.1.

$$f(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

We have $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1$. Neither of these is equal to $f(0) = 0$ and f is not continuous at 0.

Example 4.8. Consider the functions

$$f(x) = \begin{cases} \dfrac{x^3-1}{x^2-1}, & \text{if } x \neq 1 \\ 3/2, & \text{if } x = 1 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} \dfrac{x^3-1}{x^2-1}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1. \end{cases}$$

Then $\lim_{x\to 1} f(x) =$ . . . . So . . . is continuous at $x = 1$ but . . . is not continuous at $x = 1$.

Example 4.9. Consider $f(x) = \sin(1/x)$. This is defined as long as $x \neq 0$. See Figure 1 for the graph of $\sin \frac{1}{x}$.

If $c \neq 0$ then $\lim_{x\to c} \sin(1/x) = \sin(1/c)$; so f is continuous on $\mathbb{R} \setminus \{0\}$. But what happens when $x \to 0$? From the graph, we can see that we can $f(x)$ is not going to approach a particular value as $x \to 0$. So $\lim_{x\to 0} f(x)$ does not exist.

Example 4.10. Consider $f(x) = \exp(x) = e^x$. We can make e^x arbitrarily close to 0 by moving x far enough to the left. This is written $\lim_{x\to -\infty} e^x = 0$, or alternatively as $e^x \to 0$ as $x \to -\infty$. It is very important to understand that e^x never actually reaches 0. It just gets closer and closer.

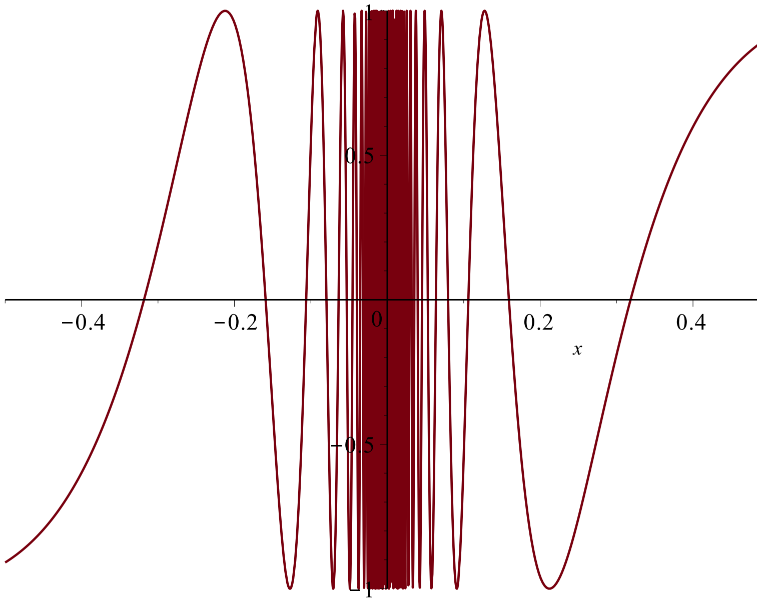


Figure 1: Graph of $\sin \frac{1}{x}$

Similarly we can consider what happens to a function as the variable becomes larger and larger. For example: $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right) =$ . . .

Limits at infinity. If $f(x)$ approaches the value ℓ as x becomes larger and larger, then we say that the limit f at x tends to ∞ is ℓ and write $\lim_{x\to\infty} f(x) = \ell$.

Similarly, $\lim_{x\to-\infty} f(x) = \ell$ means we can make $f(x)$ arbitrarily close to ℓ provided we choose x far enough left on the number line.

Note 4.11. The following observations are very useful in calculations:

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(1/x) \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f(1/x).$$

Example 4.12. Consider $f(x) = 1/x$.

Here we have:

$$\lim_{x\to-\infty} f(x) = 0, \qquad \lim_{x\to\infty} f(x) = 0,$$

$$\lim_{x\to 0^-} f(x) = -\infty, \qquad \lim_{x\to 0^+} f(x) = \infty.$$

Example 4.13. Consider $f(x) = 1+\sin(x)$. Then $\lim_{x\to\infty} f(x)$ doesn't exist and $\lim_{x\to-\infty} f(x)$ doesn't exist. However far you go to the right or the left, the function continues to oscillate between values of 0 and 2; it never settles down.

What about the function $g(x) = x\sin x$? Again, $\lim_{x\to\infty} g(x)$ does not exist. In this case, it is true that we can make $g(x)$ as large as we like by choosing suitable x far enough to the right. However, the function doesn't *stay* large - it continues to oscillate, between 0 and larger and larger values.

More examples of limit calculations.

Example 4.14. Consider the function

$$f(x) = \frac{x^3 + x - 2}{x - 1}.$$

The function f is not defined at $x = 1$ (because both numerator and denominator are zero). Let's consider some values close to $x = 1$.

x	$f(x)$
1.1	4.31
1.001	4.0030009
1.00005	4.00015
0.99	3.9701
0.9999	3.9997

It appears that $\lim_{x\to 1^+} f(x) = 4$ and $\lim_{x\to 1^-} f(x) = 4$.

Claim: $\lim_{x\to 1} \dfrac{x^3+x-2}{x-1}$ is indeed 4.

We have

$$\frac{x^3+x-2}{x-1} = \frac{(x-1)(x^2+x+2)}{x-1} = x^2+x+2 \quad \text{for } x \neq 1$$

So

$$\lim_{x\to 1} \frac{x^3+x-2}{x-1} = \lim_{x\to 1} x^2+x+2 = 4.$$

Example 4.15. Find $\lim_{x\to\infty} \dfrac{x^2+1}{3x^2+x+7}$.

$$\frac{x^2+1}{3x^2+x+7} = \frac{1+\dfrac{1}{x^2}}{3+\dfrac{1}{x}+\dfrac{7}{x^2}} \longrightarrow \frac{1+0}{3+0+0} = \frac{1}{3}.$$

Example 4.16. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

What is $\lim_{x\to 0} f(x)$? Answer: 0, because as we approach $x = 0$ from above or below the function gets arbitrarily close to 0. (It doesn't matter that this is not the value of the function there.)

Example 4.17. Consider $f(x) = x \sin\left(\frac{1}{x}\right)$. From its graph (see Figure 2), it looks like $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$. We will see a proof soon.

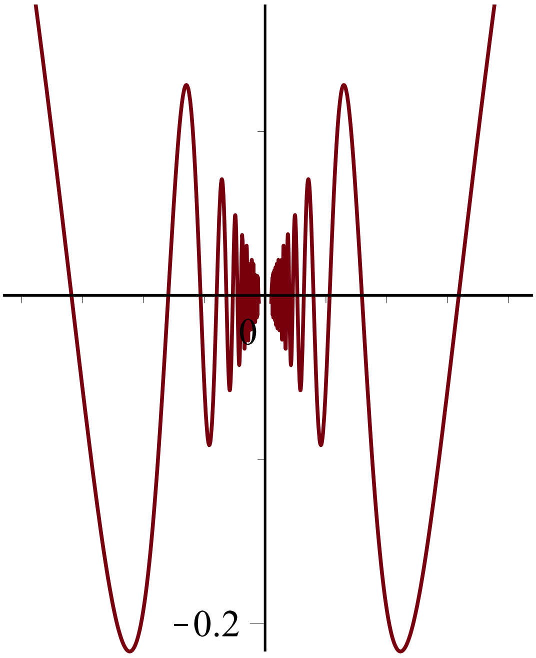


Figure 2: Graph of $x \sin \frac{1}{x}$

4.1 Limit operations

Here is a collection of basic rules for manipulating limits. We have already used some of these in examples.

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

- **Addition Rule.**

$$\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

- **Constant Rule.** For c a constant,

$$\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$$

- **Subtraction Rule** (Derived from the addition rule.)

$$\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

- **Multiplication Rule.**

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$

- **Division Rule.** If $\lim_{x\to a} g(x) \neq 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

- **Composition Rule.** If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(b) = f(\lim_{x\to a} g(x)).$$

- **Sandwich Theorem.** Suppose that $g(x) \leqslant f(x) \leqslant h(x)$ for all x in some open interval containing c (except possibly at $x = c$).
  If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then also $\lim_{x\to c} f(x) = L$.

Note 4.18. Similar rules hold for limits at $\pm\infty$ and for one-sided limits, and you are encouraged to write these out explicitly. We shall use these limit rules freely if and when needed.

Note 4.19. The rules also work for continuity of functions: *the sum, difference, product, quotient (provided the denominator is non-zero) and composite of two continuous functions is again continuous.*

We probably have used the above without really thinking about them. We shall continue to do so. (Just have a quick check to see if what you are about to conclude looks reasonable!)

Example 4.20. Since $-1 \leqslant \sin\left(\frac{1}{x}\right) \leqslant 1$, we have $-x \leqslant x \sin\left(\frac{1}{x}\right) \leqslant x$ for $x > 0$. But

$$\lim_{x \to 0^+} (-x) = 0 = \lim_{x \to 0^+} x.$$

So, by the Sandwich Theorem, $\lim\limits_{x \to 0^+} x \sin\left(\frac{1}{x}\right) = 0$.

For $x < 0$, $x \leqslant x \sin\left(\frac{1}{x}\right) \leqslant -x$, so similarly $\lim\limits_{x \to 0^-} x \sin\left(\frac{1}{x}\right) = 0$.

So $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

4.2 The limit $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

We can tabulate outputs from plugging in values of x close to 0 to get an approximation of the limit. As $(\sin x)/x$ is even, we only consider small positive values of x.

x	0.1	0.01	0.001	0.0001
$\sin x/x$	0.99833416	0.99998333	0.99999983	0.99999999

From the table and the graph (Figure 3), it looks like the limit is 1.

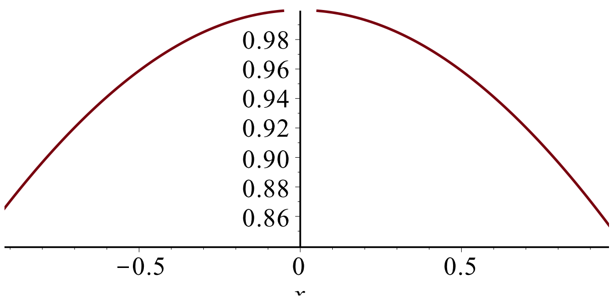


Figure 3: Graph of $\sin \frac{1}{x}$

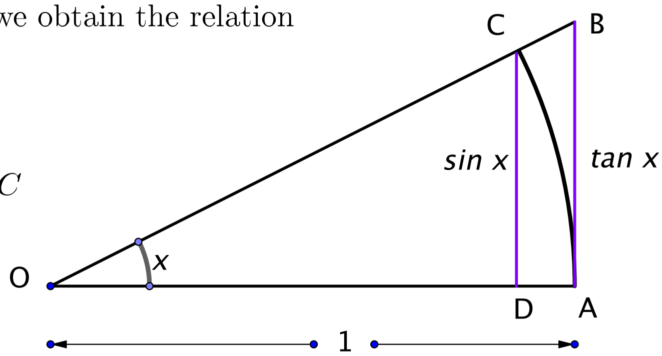
Theorem 4.21. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1.$

Using the diagaram we obtain the relation

area of ΔOCD

$\leq$ area of sector OAC

$\leq$ area of ΔOAB



This gives

Corollary 4.22. $\lim_{h \to 0} \dfrac{1 - \cos h}{h^2} = \dfrac{1}{2}$. *Thus* $\cos h \approx 1 - \frac{1}{2}h^2$ *for small* h.

Proof. $\dfrac{1 - \cos h}{h^2} = \dfrac{2\sin^2 \frac{h}{2}}{h^2} = 2\left(\dfrac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2 \to 2$ as $h \to 0$. $\square$