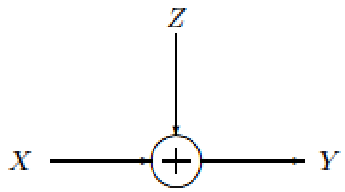
21. **A mutual information game.** Consider the following channel:



Throughout this problem we shall constrain the signal power

$$EX = 0, \qquad EX^2 = P, \tag{9.105}$$

and the noise power

$$EZ = 0, \qquad EZ^2 = N, \tag{9.106}$$

and assume that X and Z are independent. The channel capacity is given by $I(X; X + Z)$.

Now for the game. The noise player chooses a distribution on Z to minimize $I(X; X + Z)$, while the signal player chooses a distribution on X to maximize $I(X; X + Z)$.

Letting $X^* \sim \mathcal{N}(0, P), Z^* \sim \mathcal{N}(0, N)$, show that Gaussian X^* and Z^* satisfy the saddlepoint conditions

$$I(X; X + Z^*) \le I(X^*; X^* + Z^*) \le I(X^*; X^* + Z). \tag{9.107}$$

Thus

$$\begin{aligned} \min_Z \max_X I(X; X + Z) &= \max_X \min_Z I(X; X + Z) && (9.108) \\ &= \frac{1}{2} \log \left(1 + \frac{P}{N}\right), && (9.109) \end{aligned}$$

and the game has a value. In particular, a deviation from normal for either player worsens the mutual information from that player's standpoint. Can you discuss the implications of this?

Note: Part of the proof hinges on the entropy power inequality from Chapter 17, which states that if $\mathbf{X}$ and $\mathbf{Y}$ are independent random n-vectors with densities, then

$$2^{\frac{2}{n}h(\mathbf{X}+\mathbf{Y})} \geq 2^{\frac{2}{n}h(\mathbf{X})} + 2^{\frac{2}{n}h(\mathbf{Y})}. \tag{9.110}$$

Solution: *A mutual information game.*

Let X and Z be random variables with $EX = 0$, $EX^2 = P$, $EZ = 0$ and $EZ^2 = N$. Let $X^* \sim \mathcal{N}(0, P)$ and $Z^* \sim \mathcal{N}(0, N)$. Then as proved in class,

$$\begin{aligned}
I(X; X+Z^*) &= h(X+Z^*) - h(X+Z^*|X) && (9.111)\\
&= h(X+Z^*) - h(Z^*) && (9.112)\\
&\leq h(X^*+Z^*) - h(Z^*) && (9.113)\\
&= I(X^*; X^*+Z^*), && (9.114)
\end{aligned}$$

where the inequality follows from the fact that given the variance, the entropy is maximized by the normal.

To prove the other inequality, we use the entropy power inequality,

$$2^{2h(X+Z)} \leq 2^{2h(X)} + 2^{2h(Z)}. \tag{9.115}$$

Let

$$g(Z) = \frac{2^{2h(Z)}}{2\pi e}. \tag{9.116}$$

Then

$$\begin{aligned}
I(X^*; X^*+Z) &= h(X^*+Z) - h(X^*+Z|X^*) && (9.117)\\
&= h(X^*+Z) - h(Z) && (9.118)\\
&\geq \frac{1}{2}\log\left(2^{2h(X^*)} + 2^{2h(Z)}\right) - h(Z) && (9.119)\\
&= \frac{1}{2}\log\left((2\pi e)P + (2\pi e)g(Z)\right) - \frac{1}{2}\log(2\pi e)g(Z) && (9.120)\\
&= \frac{1}{2}\log\left(1 + \frac{P}{g(Z)}\right), && (9.121)
\end{aligned}$$

where the inequality follows from the entropy power inequality. Now $1 + \frac{P}{g(Z)}$ is a decreasing function of $g(Z)$, it is minimized when $g(Z)$ is maximum, which occurs when $h(Z)$ is maximized, i.e., when Z is normal. In this case, $g(Z^*) = N$ and we have the following inequality,

$$I(X^*; X^*+Z) \geq I(X^*; X^*+Z^*). \tag{9.122}$$

Combining the two inequalities, we have

$$I(X; X+Z^*) \leq I(X^*; X^*+Z^*) \leq I(X^*; X^*+Z). \tag{9.123}$$

Hence, using these inequalities, it follows directly that

$$\begin{aligned}
\min_Z \max_X I(X; X+Z) &\leq \max_X I(X; X+Z^*) && (9.124) \\
&= I(X^*; X^* + Z^*) && (9.125) \\
&= \min_Z I(X^*; X^* + Z) && (9.126) \\
&\leq \max_X \min_Z I(X^*; X^* + Z). && (9.127)
\end{aligned}$$

We have shown an inequality relationship in one direction between $\min_Z \max_X I(X; X+Z)$ and $\max_X \min_Z I(X; X+Z)$. We will now prove the inequality in the other direction is a general result for all functions of two variables.

For any function $f(a, b)$ of two variables, for all b, for any a_0,

$$f(a_0, b) \geq \min_a f(a, b). \tag{9.128}$$

Hence

$$\max_b f(a_0, b) \geq \max_b \min_a f(a, b). \tag{9.129}$$

Taking the minimum over a_0, we have

$$\min_{a_0} \max_b f(a_0, b) \geq \min_{a_0} \max_b \min_a f(a, b). \tag{9.130}$$

or

$$\min_a \max_b f(a, b) \geq \max_b \min_a f(a, b). \tag{9.131}$$

From this result,

$$\min_Z \max_X I(X; X+Z) \geq \max_X \min_Z I(X; X+Z). \tag{9.132}$$

From (9.127) and (9.132), we have

$$\begin{aligned}
\min_Z \max_X I(X; X+Z) &= \max_X \min_Z I(X; X+Z) && (9.133) \\
&= \frac{1}{2} \log \left(1 + \frac{P}{N}\right). && (9.134)
\end{aligned}$$

This inequality implies that we have a saddlepoint in the game, which is the value of the game. If signal player chooses X^*, the noise player cannot do any better than choosing Z^*. Similarly, any deviation by the signal player from X^* will make him do worse, if the noise player has chosen Z^*. Any deviation by either player will make him do worse.

Another implication of this result is that not only is the normal the best possible signal distribution, it is the worst possible noise distribution.