1 Conformal Group

Definition 1.1. The *Conformal Group* $Conf(\mathbb{R}^{p,q})$, is the connected component containing the identity in the group of conformal diffeomorphisms of the conformal compactification of $\mathbb{R}^{p,q}$.

1.1 Conformal Compactification of $\mathbb{R}^{p,q}$

Let $N^{p,q}$ be the conformal compactification of $\mathbb{R}^{p,q}$ such that for all the conformal transformations $\phi : M(\subset \mathbb{R}^{p,q}) \to \mathbb{R}^{p,q}$ there are bijective maps (which are diffeomorphic continuations of ϕ) $\hat{\phi} : N^{p,q} \to N^{p,q}$. The minimal compactification $N^{p,q}$ is equipped with the seimi-Riemannian metric.

Definition 1.2. A *semi-Riemannian* metric on a manifold M is a family of g non-degenerate bilinears such that

$$g_p : T_p \times T_p \to \mathbb{R},\ \forall p \in M \tag{1}$$

is smooth for all vector fields in M. That is all sorts of metrics we use in physics.

Definition 1.3. Projective plane $\mathbb{P}_{n+1}(\mathbb{R})$ is the extension of Euclidean plane $\mathbb{R}^{n+1}$ with the following structure

- points are added at infinity in each direction
- a line at infinity is addded which passes through all the points at infinity

Parallel lines in each direction meet at the point at infinity corresponding to that direction. Also refer Homogeneous Coordinates and Wikepedia.

Definition 1.4. The quotient map

$$\gamma : \mathbb{R}^{n+2} \mod 0 \to \mathbb{P}_{n+1}(\mathbb{R}), \tag{2}$$

maps the vector $\xi \in \mathbb{R}^{n+2}$ to a point in projection plane $\mathbb{P}_{n+1}$ given by the set of homogeneous coordinates $(\xi^0 : \ldots : \xi^{n+1})$. By the definition $\gamma(a\xi) = \gamma(\xi)\ \forall a \in \mathbb{R} \mod 0$.

Definition 1.5. Consider the map

$$\iota : \mathbb{R}^{p,q} \to \mathbb{P}_{n+1}(\mathbb{R}), \tag{3}$$

which maps the point $x \in \mathbb{R}^{p+q}$ to the point given by the homogeneous coordinates $\left(\frac{1-|x|}{2} : x^1 : \ldots : x^n : \frac{1+|x|}{2}\right)$ in $\mathbb{P}_{n+1}$ where $|x| = g_{p,q}(x,x)$ and and $p+q=n$.

Now we are in the position to identify the conformal compactification of $\mathbb{R}^{p,q}$.

Remark The closure of the image of the map ι, denoted by $\overline{\iota(\mathbb{R}^{p,q})}$ *is* $N^{p,q}$ where $N_{p,q}$ is the quadric

$$N^{p,q} := (\xi^0 : \ldots : \xi^{n+1} \in P_{n+1}(\mathbb{R}) \mid |\xi|_{p+1,q+1} = 0) \tag{4}$$

Can be seen intuitively by the definition itself (for the half of the proof). Rigorous proof not required at the moment. Note that the quadric has $p+q$ coordinates because it describes a curve on the $p+q+1$ dimensional projective space.

Now we arrive at important theorem

Theorem 1.1. *For every matrix $\Lambda \in O(p+1,q+1)$ there is a conformal map $\psi_\Lambda : N^{p,q} \to N^{p,q}$ defined by*

$$\psi_\Lambda := \gamma(\Lambda\xi)\ \forall \xi \in \mathbb{R}^{n+2} \mod 0 \mid |\xi|_{p+1,q+1} = 0 \tag{5}$$

which is conformal and diffeomorphic map. The inverse of the map $\psi^{-1} = \psi_{\Lambda^{-1}}$ is also a conformal map.

Proof. One can easily see that for all $|\xi|_{p+1,q+1} = 0$ in $\mathbb{R}^{n+2} \mod 0$, the transformation $\zeta = \Lambda\xi$ leads to $|\zeta|_{p+1,q+1} = g_{p+1,q+1}(\Lambda\xi,\Lambda\xi) = g_{p+1,q+1}(\xi,\xi) = |\xi|_{p+1,q+1} = 0$. The quotient map γ maps all such trasformed points, with zero norm, to the projective space which, by definition, is the conformal compactification of $\mathbb{R}^{p,q}$ given by $N^{p,q}$. Hence the term conformal map. □

Rest part of the theorem can be proved, but not relevant at the moment.

Remark
- Note that all the null points of $\mathbb{R}^{n+1} \mod 0$ are mapped to the null points by the matrix elements of $O(p+1,q+1)$. This is due to the fact that the metric of $\mathbb{R}^{n+2} \mod 0$ is invariant with respect to Λ.

2 Conformal Group

Definition 2.1. The conformal transformation $\phi : M \to \mathbb{R}^{p,q}$ on conntcted $M \subset \mathbb{R}^{p,q}$ has the conformal continuiation $\hat{\phi} : N^{p,q} \to N^{p,q}$ if $\hat{\phi}$ is a conformal diffeomorphism and $\iota(\phi(x)) = \hat{\phi}(\iota(x))\ \forall x \in M$.