The Eightfold Way
PHY 497 - Independent Study

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Fall 2015

Contents

1 Groups

A *group* G is a set of elements under a given operation that admit the following properties:

1. *Closure*: For all $a \in G$ and $b \in G$, we have $ab \in G$.
2. *Associativity*: For $a \in G$, $b \in G$ and $c \in G$, $(ab)c = a(bc)$.
3. *Identity*: There exists an element $e \in G$ such that for all $a \in G$, $ea = ae = a$.
4. *Inverse*: To every element $a \in G$, an inverse element $a^{-1} \in G$ exists such that
$$a^{-1}a = aa^{-1} = e.$$

We will not expand on the many theorems and definitions of group theory, but rather examine the applications of their main ideas to particle physics. In essence, we use the properties of certain groups to explain the physical phenomenon, without examining elements of the group in detail.

We are primarily interested in groups of permutations and groups of matrices. The group of permutations on a finite set of n objects is called the *symmetric group*, denoted S_n, and the group operation is composition. Later, we will find that eigenstates of multiple particle systems can be written in terms of symmetric and anti-symmetric permutations on the particles.

The *general linear group of degree* n is the set of all invertible $n \times n$ matrices, forming a group together under the operation of ordinary matrix multiplication. We take the entries of these matrices to be complex-valued, and denote this group $GL(n, \mathbb{C})$. Many matrices belong to this group, but several important subgroups exist. Of paramount importance to particle physics are unitary matrices (a particular of the more general Hermitian conjugate). A *unitary* matrix is a square matrix such that its conjugate transpose is its inverse, that is,

$$U^*U = UU^* = I$$

with * denoting the conjugate transpose and I the identity matrix. Unitary matrices (or equivalently as operators) remarkably preserve norms, and thus probability amplitudes. In other words, the wavefunction is invariant under a unitary matrix. The set of all $n \times n$ unitary matrices form a group under multiplication, denoted $U(n)$, and those matrices that have determinant 1 form the *special unitary group* $SU(n)$. Notice that $|\det U| = 1$ for all unitary matrices, whereas a matrix with real-valued determinant 1 belongs to $SU(n)$.

2 Group Representations

In particle physics, we deal more with the representations of a group than elements of the group itself. A *representation* of a group G is a homomorphism of G onto a group of linear operators acting on a linear vector space. If the linear operators are taken to be matrices, then we call the representation a *matrix representation.* For the purposes of this paper, we will always mean a matrix representation.

A representation D on a group G (assumed to be over the field $\mathbb{C}$) is a map

$$D : G \to GL(V)$$

such that

$$D(g_1 g_2) = D(g_1)D(g_2) \quad \forall g_1, g_2 \in G$$

Here, the dimension of V is the dimension of the representation. If V is of finite-dimension n, then typically we identify $GL(V)$ with $GL(n, \mathbb{C})$. Denote the set of all matrices of a representation of G by $D(G)$. If we consider more than one representation, distinguish between them with a superscript $D^{(i)}(G)$.

Now consider four representations of a two element group $G = \{e, a\}$.

1. $D^{(1)}(e) = 1, \qquad D^{(1)}(a) = -1,$
2. $D^{(2)}(e) = 1, \qquad D^{(2)}(a) = 1,$
3. $D^{(3)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^{(3)}(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$
4. $D^{(4)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D^{(4)}(a) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$

Notice that $D^{(4)}$ has square matrices with vanishing determinants. Hence, these matrices do not belong to $GL(2, \mathbb{C})$ and are not useful. Also, the representations $D^{(1)}, D^{(3)}$, and $D^{(4)}$ are an *isomorphism* (a homomorphism that is one-to-one) of G, while $D^{(2)}$ is not. A representation that is isomorphic to the group is called a *faithful* representation. It is clear that all matrix groups are faithful representations of themselves.

The trace of a matrix $D(g) \in D(G)$ is called the *character* of g, denoted by $\chi(g)$. The set of all characters of elements is called the character of the representation, denoted by $\chi(D)$. The importance is that *all equivalent representations have the same character.* This enables us to say, for instance, that $D^{(1)}$ and $D^{(4)}$ are equivalent representations, since $\chi(D^{(1)}) = \{1, -1\} = \chi(D^{(4)})$.

In general, a representation D is said to be decomposable into a direct sum of representations $D^{(i)}$ if we can write

$$D = D^{(1)} \oplus D^{(2)} \oplus \cdots \oplus D^{(k)},$$

where the $\oplus$ operation takes matrices $D^{(i)}$ and arranges them as a block matrix with matrices on the diagonal and zeroes everywhere else, that is,

$$D = \begin{pmatrix} D^{(1)} & 0 & \cdots & 0 \\ 0 & D^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^{(k)} \end{pmatrix}.$$

If a representation $D^{(i)}(G) : G \to GL(V)$ is restricted to an invariant subspace $W \subset V$ from the group, $D^{(i)}(G)$ is a subrepresentation of V. If a representation $D^{(i)}$ has only trivial invariant subspaces, that is, $\{0\}$ and V itself, then $D^{(i)}$ is said to be an *irreducible representation.*

The use of irreducible representations in particle physics is widespread and explored in the next section.

3 Unitary Representations and Multiplets

3.1 Motivation for Unitary Representations

As stated before in Section 1, unitary matrices preserve norms. More precisely, given two complex vectors x and y, multiplication by U preserves their inner product,

$$\langle Ux, Uy \rangle = \langle x, y \rangle.$$

Implications of this are important for energy considerations. The Hamiltonian operator H describes the energy of a system, and is always a real number. The spectrum of allowed energy levels of the system is given by a set of eigenvalues E_n and solve the equation

$$H\psi_n = E_n \psi_n$$

If we operate on this equation with a unitary operator U, we obtain

$$UH\psi_n = UHU^{-1}U\psi_n = E_n U\psi_n.$$

Letting

$$H' = UHU^{-1} \qquad \text{and} \qquad \psi'_n = U\psi_n,$$

we get

$$H'\psi'_n = E_n\psi'_n.$$

By assumption, the transformation U leaves H invariant, which implies $H' = H$. Then, we can say ψ'_n is an eigenstate of the Hamiltonian with the same energy as ψ_n. Continuing to act on the system with another unitary transformation, we will find another eigenstate of H with the same energy eigenvalue E_n.

All states obtained in this way can be written as a linear combination of basis vectors of the unitary representation of the group of transformations. In general, the vectors form a basis of an irreducible representation.

3.2 Multiplets

The set of basis vectors of an irreducible unitary representation (of transformations) denote a set of quantum mechanical states. Call this set of states a **multiplet**. Since all of the states of a multiplet are eigenstates of the Hamiltonian with the same energy eigenvalue, the states are said to be *degenerate* in the energy.

As an example, the different charge states of a particle with isospin I constitute a multiplet. But since isospin is not an exact symmetry, the states are not exactly degenerate. For ease then, different states are referred to as distinct particles, rather than substates of the same particle.

Now, we identify the members of multiplets (states given by $|j, m\rangle$) from the combination of two spin-$\frac{1}{2}$ particles. They can have total spin $j = 0$ or $j = 1$ with $m \in \{-1, 0, 1\}$.

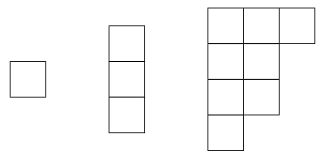
$$\begin{aligned}
|1,1\rangle &= |\tfrac{1}{2},\tfrac{1}{2}\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle && = |\uparrow,\uparrow\rangle \\
|1,0\rangle &= \tfrac{1}{\sqrt{2}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{2}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle && = \tfrac{1}{\sqrt{2}}\,|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle \\
|1,-1\rangle &= |\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle && = |\downarrow,\downarrow\rangle \\
|0,0\rangle &= \tfrac{1}{\sqrt{2}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{2}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle && = \tfrac{1}{\sqrt{2}}\,|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle
\end{aligned}$$

Here, the set of states with $j = 1$ form a triplet that is symmetric under the interchange of particles. The singlet contains one $j = 0$ which is deduced from requiring orthogonality to the $|0, 1\rangle$ state, and is totally antisymmetric.

In general, this process for describing multiplets is longwinded and tedious, which may or may not include orthogonality considerations. We are then seeking an efficient way to describe multiplets of particle combinations.

4 Young Tableaux

A Young tableaux is a combinatorial object that provides a convenient way to describe the number, dimension, and symmetries of irreducible representations.



The diagram itself is a collection of rows of boxes, left justified. The i-th row has λ_i boxes and is constrained such that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n.$$

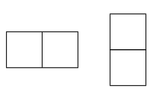
This means no row may exceed the number of boxes in the previous row. The same constraint applies for columns.

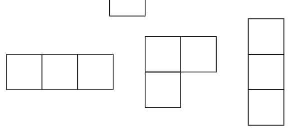
4.1 Tableaux for the Symmetric Group

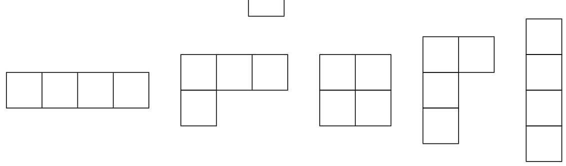
When discussing the symmetric group, S_n, we also have the requirement that

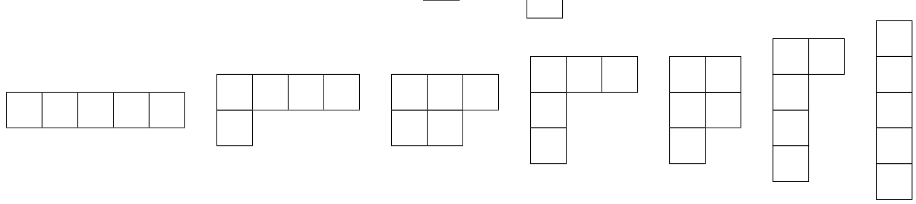
$$\sum_{i=1}^{n} \lambda_i = n.$$

Each tableaux is then a partition of the integer n. By exhausting all possibilities of arranging n boxes given the row constraints, we find:

S_2 : 

S_3 : 

S_4 : 

S_5 : 

and so on.

Define the *hook-length* of a particular box in a Young diagram to be the number of all boxes to the right, plus one for the box itself, plus the number of all boxes in the column below it. Also, call the product of all hook-lengths of a diagram $\mathcal{N}$. The hook-lengths of the tableaux in S_4 are displayed below:

$$
\begin{array}{llllll}
S_4: & \begin{array}{|c|c|c|c|}\hline 4&3&2&1\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 4&2&1\\\hline 1\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 3&2\\\hline 2&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 4&1\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array} & \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \\
& & & & & \\
\mathcal{N}: & 24 & 8 & 12 & 8 & 24
\end{array}
$$

This is precisely what we need to calculate the dimension of an irreducible representation, $Y(\lambda) : S_n \to GL(V)$, where λ is the shape of a tableaux. It is given by the following formula:

$$\dim Y(\lambda) = \frac{n!}{\mathcal{N}}.$$

Thus, from above with S_4, we get

$$
\begin{aligned}
\dim \begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline\end{array} &= 4!/24 &&= 1\\
\dim \begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline \ \\\cline{1-1}\end{array} &= 4!/8 &&= 3\\
\dim \begin{array}{|c|c|}\hline \ &\ \\\hline \ &\ \\\hline\end{array} &= 4!/12 &&= 2\\
\dim \begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1} \ \\\cline{1-1}\end{array} &= 4!/8 &&= 3\\
\dim \begin{array}{|c|}\hline \ \\\hline \ \\\hline \ \\\hline \ \\\hline\end{array} &= 4!/24 &&= 1
\end{aligned}
$$

The dimension of an irreducible representation found using the hook-length formula goes beyond the symmetric group. We see next how the tableaux are applied to the special unitary groups, which are far more important to the physics.

4.2 Tableaux for the Special Unitary Group

Now, our tableaux operate under slightly different rules for $SU(n)$ than S_n. The most striking difference is that the number of boxes need not be n, and in fact, the fundamental unit is a single box (e.g. particle) that can take n labels.

$$SU(3) : \boxed{} = \boxed{1} \text{ or } \boxed{2} \text{ or } \boxed{3}$$

We can also begin to classify symmetry, such that horizontal rows indicate symmetric combinations, vertical rows indicate antisymmetric combinations, and all other configurations are mixed:

$$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array} \text{ symmetric } \quad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array} \text{ antisymmetric } \quad \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \text{ mixed}$$

When populating the boxes with particle labels (1,2,3 etc...), several rules are in place to avoid double-counting permutations:

1. Particle numbers may not decrease along the row.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline \end{array} \text{ is allowed but } \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 2 \\ \hline \end{array} \text{ is not.}$$

2. Particle numbers must strictly increase down the column.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \text{ is allowed but } \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \text{ is not.}$$

3. Tableaux for $SU(n)$ have at most n rows. This follows from (2).

Calculating the dimension of tableaux in $SU(n)$ obeys a slightly different counting rule. First, insert n into the top left box of the tableaux. Then, strictly increase n by single increments across the row. Strictly decrease n by single increments down the column, and proceed until all boxes have been filled. An example of this counting scheme for $SU(4)$ is given below:

$$\begin{array}{|c|c|c|c|}\hline 4 & 5 & 6 & 7 \\ \hline 3 & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Call the product of all numbers in the boxes $\mathcal{D}$. Similar to S_n, we will also inherit the product of hook-lengths $\mathcal{N}$. Then, the dimension of an irreducible representation $Y(\lambda) : SU(n) \to GL(V)$ is given by

$$\dim Y(\lambda) = \frac{\mathcal{D}}{\mathcal{N}}$$

The following tableaux will be used frequently for our discussion of particle interactions, so we calculate the dimensions now:

In $SU(2)$:

$$
\begin{aligned}
\dim \ydiagram{1} &= 2/1 = 2 \\
\dim \ydiagram{2} &= 2 \cdot 3/2 \cdot 1 = 3 \\
\dim \ydiagram{3} &= 2 \cdot 3 \cdot 4/3 \cdot 2 \cdot 1 = 4 \\
\dim \ydiagram{1,1} &= 2 \cdot 1/2 \cdot 1 = 1
\end{aligned}
$$

In $SU(3)$:

$$
\begin{aligned}
&\dim \ydiagram{1} = 3/1 = 3 \\
&\dim \ydiagram{2} = 3 \cdot 4/2 \cdot 1 = 6 \\
&\dim \ydiagram{3} = 3 \cdot 4 \cdot 5/3 \cdot 2 \cdot 1 = 10 \\
&\dim \ydiagram{2,1} = 3 \cdot 4 \cdot 2/3 \cdot 1 \cdot 1 = 8 \\
&\dim \ydiagram{4,2} = 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2 \cdot 3/5 \cdot 4 \cdot 2 \cdot 1 \cdot 2 \cdot 1 = 27 \\
&\dim \ydiagram{1,1,1} = 3 \cdot 2 \cdot 1/3 \cdot 2 \cdot 1 = 1
\end{aligned}
$$

However, it is likely that we will run into tableaux with the same dimension, but different overall symmetries. For example, in $SU(3)$, notice that

$$\dim \ydiagram{3} = \dim \ydiagram{3,3} = 10.$$

In this scenario, we say that these two are *conjugate tableaux*. It may also arise that a tableaux is *self-conjugate*, but if a tableau and its conjugate are *not* the same, then they represent irreducible tensors of inequivalent representations with the same dimensionality. Thus, in our example, the tableaux have different diagrams and are of course inequivalent. We say then that

$$
\begin{aligned}
\dim \ydiagram{3} &= 10 \\
\dim \ydiagram{3,3} &= \overline{10}
\end{aligned}
$$

in order to distinguish the two. The choice of the overline is arbitrary.

4.3 Operations on Young Tableaux

The significant advantage of using tableaux is being able to describe all the states of a system without tediously finding linearly independent basis tensors. While the process described in Section 3.2 is manageable (and instructive) for low-dimensions and few-particle systems, Young Tableaux in $SU(n)$ reduces this process to simple and insightful calculations.

Say we wish to study the representations of multi-particle systems. Each present particle belongs to a particular representation that describes its state. The set of all states forms a multiplet. The product of multiplet representations will then describe the states of the multi-particle system. In other words, the direct product of representations decomposes into a direct sum of irreducible representations:

$$\bigotimes_{i=1}^{m} D^i = \bigoplus_{k=1}^{n} D^k$$

This is an amazing fact! Because in general, the Kronecker product on two matrices takes $A = m \times n$ and $B = p \times q$ and creates $A \otimes B = mp \times nq$. This can create a tremendously large matrix; however, by the right hand side of the equation above, it decomposes into a simpler block matrix. Thus, when we act on the system with a product of representations, only the relevant irreducible representations take part.

However, we must define the product of two Young Tableaux in order to utilize them to this effect.

Start with a very straightforward example. We know from Section 3 the combination of two spin-$\frac{1}{2}$ particles forms a symmetric triplet and a totally antisymmetric singlet. The combination of base states is exactly our Kronecker product, and the resulting multiplets are in a direct sum. Thus, we get

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$$

If we consider these particles to exhibit $SU(2)$ symmetry, then we have already seen tableaux in $SU(2)$ with these dimensions. Then we can say

$$\square \otimes \square = \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$$

Recall that a totally horizontal tableaux is symmetric, and a totally vertical tableaux is antisymmetric, and this fits precisely with our observation. But, notice that if we chose our group to be $SU(3)$, then the *same* tableaux equation gives

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}.$$

The equation also hints at the process to multiply tableaux.

4.4 Rules for Multiplying Tableaux

1. Each tableau should be a proper: no row is longer than the row above it.
2. No tableau should have a column with more than n boxes if the group is $SU(n)$. One can strike out any column with exactly n boxes, as this does not affect dimensionality.
3. Label numbers must not decrease along the row.
4. Label numbers must increase down the column.
5. Count each row starting from the rightmost box of the topmost row. If the number of boxes with a label number i exceeds the number of encountered boxes with label $i-1$ at any point during our count, then the tableau is omitted from the decomposition entirely.

It is sufficient to only label the second term in the products. Simply insert 1's into the top row boxes, 2's in the second row, and so on.

Next, we follow the rules and use multiplication rules to compute the combination of three particles in $SU(2)$.

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \left(\ydiagram{2} \oplus \ydiagram{1,1} \right) \otimes \ytableaushort{1}$$

$$= \ytableaushort{{}{}1} \oplus \ytableaushort{{}{},1} \oplus \ytableaushort{{}1,{}}$$

$$= \ydiagram{3} \oplus \ydiagram{1} \oplus \ydiagram{1}$$

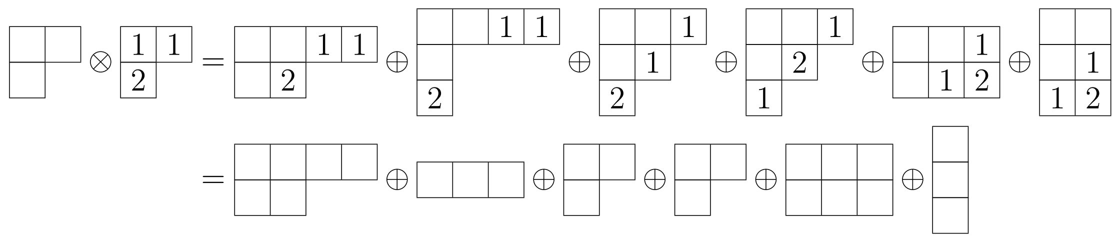
$$\Rightarrow \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}.$$

Notice that when we switch to $SU(3)$, we get the same decomposition, only now the totally antisymmetric tableaux $\ydiagram{1,1,1}$ appears and the two-box columns of the mixed symmetry are not omitted.

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

$$\Rightarrow \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$$

The last decomposition we work out by hand is the product of two eight-dimensional multiplets. This has significance, described later.



$$\Rightarrow \mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \overline{\mathbf{10}} \oplus \mathbf{1}.$$

5 Quarks

Interestingly, the behavior of elementary particles was proposed based on the behavior of special unitary group (in actuality, the behavior of the Lie algebra corresponding to $SU(n)$, though not explored in this paper). Then, when considering the theory of strong interactions, we say particles possess *color charge*, the strong-analog of electric charge in electromagnetic interactions. From here on out, we belong to $SU(3)$.

The particles of consideration are called *quarks*, and quarks combine to form composite particles called hadrons, and the hadrons can be observed experimentally. Examples of hadrons include protons, neutrons, pions, etc. and interact via the strong interaction. There are two main families:

1. Baryons − particles composed of three quarks and possess half-integral spin. All except the proton is unstable.

2. Mesons − particles composed of a quark−antiquark pair and have integral spin. All are unstable.

We say quarks can have either red, blue, or green charge. Antiquarks possess antired, antiblue, antigreen charge. The theory of quarks in the strong interaction is motivated by symmetry and quantum labelling of states. The advantage of using group theory in particle physics is being able to classify all possible hadrons (due to legal quark combinations) into these multiplets.

5.1 Quantum Numbers

5.1.1 Isospin and Hypercharge

The quantum number of isospin I is believed to be conserved in strong interactions. It can be realized in an abstract three-dimensional space, analagous to total angular momentum J in ordinary space. It has components I_1, I_2, and I_3. We say I_3 is related to the charge of a strongly interacting particle, thus, rotations around the first or second axis in isospin space *changes the charge of a state*. More formally, I_3 is the eigenvalue of the projection of I_z.

For example, Heisenberg suggested in 1932 that neutron and proton might be treated as different charge substates of one particle, the nucleon. A nucleon is said to have $I = \frac{1}{2}$ and two substates with $I_3 = \pm\frac{1}{2}$. By convention, we assign a proton to $I_3 = +\frac{1}{2}$ and a neutron to $I_3 = -\frac{1}{2}$. Assigning isospin to particles is fully determined with the addition of another quantum number, Y.

We also believe hypercharge Y is conserved. There is a relation between the electric charge of a state Q (in units of the proton charge e) and the isospin. For all members of a given isospin multiplet, Q and I_3 are related by

$$Q = I_3 + Y/2.$$

This is the Gell-Mann−Nishijima formula. Mathematically, the multiplets of $SU(2)$ are the isospin multiplets (such as proton and neutron), and the hypercharge is the generator of $U(1)$. The group product $SU(2) \times U(1)$ can model the symmetry breaking of the electroweak theory, though not explored fully here.

5.1.2 Baryon Number and Strangeness

Hadrons have baryon number B. The strangeness S of a hadron is given by

$$S = Y - B.$$

Quarks individually possess fractional baryon number $B = \frac{1}{3}$, and the combination of three quarks form a baryon with $B = 1$.

Antiparticles carry the same fractional baryon number, only with negative sign. The combination of a quark and antiquark with $B = \frac{1}{3}$ and $B = -\frac{1}{3}$ respectively form the meson with $B = 0$.

It is experimentally shown that the baryon states can be arranged systematically in multiplets of fixed spin and parity, J^P. We distinguish between baryon states by requiring that constituent quarks have different *flavors*. In $SU(3)$ multiplets,

these consist of *up, down* and *strange* quarks (or red, green, and blue). Different combinations of u, d and s yield different baryons, and the quantum numbers of these quarks are summarized in the table below.
Antiquarks $\overline{u}, \overline{d}$, and $\overline{s}$ have the signs of B, I_3, S, and Q reversed.

Table 5.1. Quark quantum numbers

Flavor	B	J	I	I_3	S	Q
u	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	0	$+\frac{2}{3}$
d	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$
s	$\frac{1}{3}$	$\frac{1}{2}$	0	0	-1	$-\frac{1}{3}$

From here we can build all the baryons that we will soon find within octets and decuplets predicted by quark interactions.
This leads directly into our discussion of the $SU(3)$ multiplets which inspired the

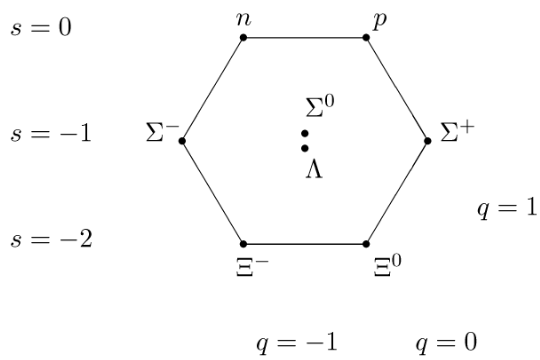
Table 5.2. Baryons

	I	Y
N	$\frac{1}{2}$	1
Λ	0	0
Σ	1	0
Ξ	$\frac{1}{2}$	-1
Δ	$\frac{3}{2}$	1
Ω	0	-2

quantum number scheme.

5.2 The Baryon Octet

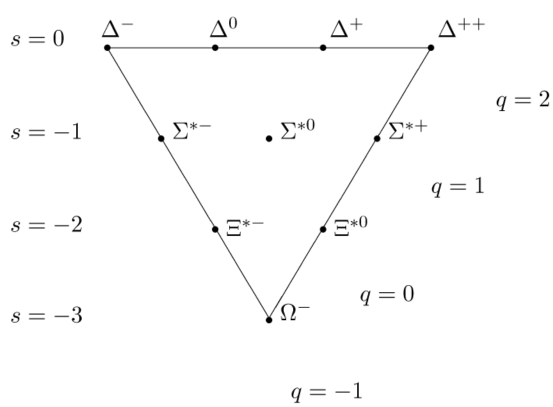
The baryons with spin-parity $J^P = \frac{1}{2}^+$ form an octet. This can be realized as all the combinations of up, down, and strange quarks with the proper spin-parity. If we plot Y vs. I_3, we can plot the eight hadrons in an insightful way.



There is another octet consisting of the antiparticles of the baryon octet. This is the baryons with $J^P = \frac{1}{2}^-$ and $B = -1$. In general, the antiparticles belong precisely to the conjugate representation, which in the case of $SU(3)$, has the octet self-conjugate.

5.3 The Baryon Decuplet

The baryons with spin-parity $J^P = \frac{3}{2}^+$ form a decuplet. This classification is remarkable in that in 1962 it predicted the existence of the three-strange particle Ω^- before being experimentally found in 1964.



Likewise, the antiparticles would belong to the decuplet conjugate found in $SU(3)$ multiplets. It so happens that $\Sigma-$ and $\Xi-$particles appear in *both* the octet and decuplet, but we use the asterisk in the decuplet, since these particles with $J^P = \frac{3}{2}^+$ are in the excited state.

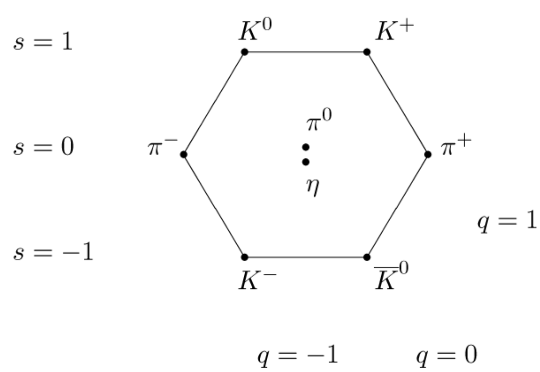
5.4 Mesons

Recall that mesons have $B = 0$, and have integral spin. Meson multiplets of $SU(3)$ differ from baryon multiplets in several ways. Thus far, all notable observations of mesons belong to either a singlet or an octet.

Table 5.2. Mesons

	I	Y
π	1	0
K	$\frac{1}{2}$	1
η	0	0
η'	0	0

Also, unlike baryons, the meson's particle-antiparticle pair have the same values of quantum numbers, namely spin, parity, and baryon number. Thus, for every state in a multiplet (either the same state or another), there is a state with the quantum numbers of the antiparticle. The following diagram is for the $J^P = 0^-$ octet.



Note: η' is a singlet state not depicted in this octet

A third difference actually challenges the symmetries of $SU(3)$. With mesons, if two multiplets exist with the same values of spin, parity, and baryon number, then a symmetry-breaking interaction can "mix" the two multiplets. An example is the $I = Y = 0$ singlet state and the $I = Y = 0$ octet state. Then it is possible for the symmetry-breaking interaction to cause the two physical states to be linear combinations of the singlet and octet!

$SU(3)$ mixing occurs to a higher degree in mesons than baryons, so the octet and singlet is often combined into a nonet, but such a nonet is reducible in $SU(3)$. We will not further discuss mesons, as the notation and content in literature is even more disorganized than baryons!

6 The Eightfold Way and Color Confinement

The motivation for classifying hadrons into an octet and decuplet is apparent from the decomposition of particle combinations aided by Young Tableaux.

Combining three quarks is analogous to multiplying 3-dimensional representations of $SU(3)$ multiplets.

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$$

Contained in the decomposition is our familiar octet **8** and decuplet **10**! However, this is a disadvantage as it does not contain the antiparticle decuplet. To move around this, Murray Gell-Mann utilized the lowest dimensional representation of $SU(3)$ that possesses *zero triality*.

6.1 Triality

We seek to understand the class of a representation. We subdivide the the representations of $SU(n)$ into n classes corresponding to the number of boxes in their Young Tableaux. The number of boxes v of any tableau is given by

$$v = ni + k$$

where $i, k \in \mathbb{Z}$ such that $i \geq 1$ and $0 \leq k \leq n - 1$. The remainder k describes a specific class of representation. If we decompose the product of two representations of classes k_1 and k_2, then the irreducible representations contained in the product are of class k where

$$k \equiv k_1 + k_2 \mod n.$$

In $SU(3)$, the possible remainders are $k = 0, 1,$ or 2 and we say the *triality* τ of a representation is

$$\begin{aligned} \tau &= 0 \quad \text{if} \quad k = 0, \\ \tau &= 1 \quad \text{if} \quad k = 1, \\ \tau &= -1 \quad \text{if} \quad k = 2. \end{aligned}$$

Then, the lowest dimensional representation with zero-triality has a Young Tableaux with three boxes, and on inspection we find that this is $\ydiagram{2,1}$, or $\mathbf{8}$. And all the zero-triality representations of $SU(3)$ can be obtained from the product of two $\mathbf{8}'s$, by above. Thus, the $\mathbf{8}$, rather than the $\mathbf{3}$, is intrinsic to this approach, leading Gell-Mann to dub this symmetry scheme as "the eightfold way":

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \overline{\mathbf{10}} \oplus \mathbf{1}.$$

6.2 Confinement

We require that no free quarks exist, since a free quark would carry a color charge. Consequently, quarks are always found in colorless (white) bound states. We come upon our **color confinement hypothesis:** *only color singlet states exist as free particles.*

This is implied by the decomposition of irreducible representations. If a particle combination does not contain a singlet (totally antisymmetric state), then bound states of the particle interactions do not exist. We have seen this before. Two particle interactions in $SU(3)$ have the decomposition $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}$. But when we consider quark combinations, two-quark (qq) combinations would have color charge, impossible by color confinement hypothesis. But, as illustrated in Section 6, three-quark combinations (qqq) contains a singlet state $\mathbf{1}$, and hence bound states exist. In fact, the wavefunction for baryons is

$$\psi^{qqq} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$$

This satisfies a requirement that raising and lowering operators (operators that exchange particles) yield zero on this wavefunction. Also, the quantum numbers describing this state is $I_3 = Y = 0$, while the system is invariant under other $SU(3)$ color combinations.

Likewise, for the same reasons, the wavefunction for mesons is

$$\psi^{q\overline{q}} = \frac{1}{\sqrt{3}}(u\overline{u} + d\overline{d} + s\overline{s}).$$

It is important to realize that the quark formations determine the hadron, but the members of the octet or decuplet are described by a gauge field theory. According to this, strong interaction arises from the $SU(3)$ symmetry among quarks. Then we introduce the so-called "gluon", a virtual massless exchange particle that mediates strong interactions. Gluons carry color and anticolor ($q\overline{q}$) and thus obey the some color wavefunctions as mesons in an octet and colorless singlet: $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

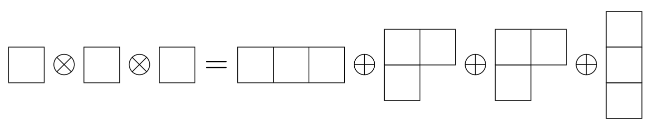
The product of **8**'s in the Eightfold Way is then realized as gluon-pair interactions. Gluons can then be in a variety of color states, including a singlet, which explains the observability of the decuplet and octet.

Technically, the gluons are vector fields in the adjoint representation of $SU(3)$. In general, the number of carriers is always equal to $n^2 - 1$ for $SU(n)$, so for us, the gluons belong to the octet. This is perhaps a more intuitive, yet difficult, way of describing the gluon's importance independent of triality. Further discussions on gluon interaction deviate away from group-theoretical methods and are beyond the scope of this paper.

7 Other Models and Experimental Evidence

7.1 SU(6)

The classification of hadrons arose from $SU(3)$ exact symmetry, but since particles have different mass energies, the symmetry of $SU(3)$ is badly broken. A similar model considers $SU(6)$ with up, down, strange, top, bottom, and charm quarks, but the symmetry here is broken as well. Interestingly enough, in $SU(6)$, we get



$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56} \oplus \mathbf{70} \oplus \mathbf{70} \oplus \mathbf{20}$$

for the combination of three particles.

If spin is included with flavor, in the totally symmetric 56-plet, we have

$$56 = (2, 8) + (4, 10)$$

where the first number in parentheses refers to the spin multiplicity $2J + 1$ and the second is an $SU(3)$ multiplet. Thus, the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet appears in higher symmetry groups; namely when the three particles are u, d, and s quarks.

7.2 Cross-sections of Pion-Nucleon Scattering

The concepts developed in this paper only scratch the surface of the amount needed to predict particle interactions. However, the quark model does make predictions on the relative magnitudes of high-energy hadron-hadron interactions.

We are primarily interested in the cross-section σ of elastic scattering of the pion and nucleon, measured in millibarns mb. At high energy, $\sigma(qq) = \sigma(q\bar{q})$ and there is isospin invariance for $u-$ and $d-$cross sections: $\sigma(uu) = \sigma(dd) = \sigma(ud)$. Then, from counting residual quarks after a pion-nucleon scattering,

$$\frac{\sigma(\pi N)}{\sigma(NN)} = \frac{2}{3}.$$

This is confirmed experimentally! When the energy of the pion is 60 GeV, $\sigma(\pi^- p) = \sigma(\pi^+ p) = 24$ mb, while $\sigma(pp) = \sigma(pn) = 38$ mb. Thus, the ratio is approximately $\frac{2}{3}$.

Also, we have

$$\frac{\sigma(\pi^- p)}{\sigma(\pi^+ p)} = \frac{1}{3}.$$

Data shows that $\sigma(\pi^- p) \approx 70$ mb and $\sigma(\pi^+ p) \approx 200$ mb at 1.2 GeV, a ratio with an error of approximately 5%. Thus, with a simple quark model influenced by group theory, we are able to make astounding predictions about strong interactions and verify these claims experimentally.

8 References

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