1. After omitting the constant λk from the primal objective, using variables α_j and $\beta_{i,j}$ the dual becomes

$$\max \sum_{j\in C} \alpha_j$$

s.t.

$$\begin{aligned}
\alpha_j - \beta_{i,j} &\leq c((i,j)) && \forall i \in F, \forall j \in C \\
\sum_{j\in C} \beta_{i,j} &\leq \lambda && \forall i \in F \\
\alpha_j &\geq 0 && \forall j \in C \\
\beta_{i,j} &\geq 0 && \forall i \in F, \forall j \in C
\end{aligned}$$

2. Since $\forall f_{i_C} = \lambda$, the formula becomes

$$3|S|\lambda + \sum_{Cluster C_0} \sum_{j\in C_0} c((i_{C_0}, j)) \leq 3 \sum_{j\in C} \alpha_j$$

.

3. The cost of the solution (LP1) and the dual value has the following relationship: $\sum_{Cluster C_0} \sum_{j\in C_0} c((i_{C_0}, j)) \leq 3(\sum_{j\in C} \alpha_j - |S|\lambda)$.

4. In case of $\lambda_1 = 0$ there is no cost for opening a facility, therefore the solution is to open every facilities and assign each clients to (one of) the closest facilities. If $|F| \geq k$, then the algorithm opens at least k facilities. When using λ_2, opening a facility is more costly than using every possible edge in the graph. Therefore the algorithm opens only one facility, which has the lowest total distance from the clients.

5.

$$cost(S_1) = \sum_{i\in S_1} \sum_{j\in Cluster(i)} c((i,j)) \leq 3(\sum_{j\in C} (\alpha_1)_j - |S_1|\lambda_1)$$

and

$$cost(S_2) = \sum_{i\in S_2} \sum_{j\in Cluster(i)} c((i,j)) \leq 3(\sum_{j\in C} (\alpha_2)_j - |S_2|\lambda_2).$$

6. $cost(S_1) \leq 3(\sum_{j\in C}(\alpha_1)_j - |S_1|\lambda_1) \leq 3(\sum_{j\in C}(\alpha_1)_j - |S_1|(\lambda_2 - \frac{\epsilon c_{\min}}{3|F|}))$
$= 3(\sum_{j\in C}(\alpha_1)_j - |S_1|\lambda_2) + \frac{|S_1|\epsilon c_{\min}}{|F|} \leq 3(\sum_{j\in C}(\alpha_1)_j - |S_1|\lambda_2) + \epsilon\text{OPT}.$

In the last inequality, we have used that $c_{\min} \leq \text{OPT}$ and $|S_1| \leq |F|$.

7. Using 5 and 6:

$$\delta_1 cost(S_1)+\delta_2 cost(S_2) \leq 3\delta_1(\sum_{j\in C}(\alpha_1)_j - |S_1|\lambda_2)+\delta_1\epsilon\text{OPT}+3\delta_2(\sum_{j\in C}(\alpha_2)_j - |S_2|\lambda_2)$$

$$= 3(\delta_1\sum_{j\in C}(\alpha_1)_j + \delta_2\sum_{j\in C}(\alpha_2)_j - \lambda_2(\delta_1|S_1| + \delta_2|S_2|)) + \delta_1\epsilon\text{OPT}.$$

Since $\delta_1|S_1| + \delta_2|S_2| = k$ and $\delta_1\sum_{j\in C}(\alpha_1)_j + \delta_2\sum_{j\in C}(\alpha_2)_j = \sum_{j\in C}\widetilde{\alpha}_j \leq \text{OPT}$, we get

$$\delta_1 cost(S_1) + \delta_2 cost(S_2) \leq 3(\text{OPT} - \lambda_2 k) + \delta_1\epsilon\text{OPT}$$
$$\leq 3\text{OPT} + \delta_1\epsilon\text{OPT} = (3 + \delta_1\epsilon)\text{OPT}.$$

8. If $\delta_2 \geq \frac{1}{2}$ then

$$\frac{1}{2}cost(S_2) \leq \delta_2 cost(S_2) \leq \delta_1 cost(S_1)+\delta_2 cost(S_2) \leq (3+\delta_1\epsilon)\text{OPT} \leq (3+\epsilon)\text{OPT}.$$

Therefore $cost(S_2) \leq 2(3 + \epsilon)\text{OPT}$.

9. The algorithm chooses randomly $k-|S_2|$ facilities from the possible $|S_1|-|S_2|$ facilities not yet opened from S_1. Therefore the probability of opening a facility randomly is $\frac{k-|S_2|}{|S_1|-|S_2|} = \delta_1$.

10. Since the algorithm opens the facility from S_1 that is the closest to f_2, which was i and not f_1, thus $c((i, f_2)) \leq c((f_1, f_2))$.

11. $c((i,j)) \leq c((j, f_2)) + c((f_2, i)) \leq c((j, f_2)) + c((f_1, f_2)) \leq c((j, f_2)) + c((f_1, j)) + c((j, f_2)) = c_j^1 + 2c_j^2$, where the first and last inequalities come from the triangle inequality, while the second inequality comes from 10.

12. With probability δ_1 the facility f_1 is open and the cost becomes c_j^1. Otherwise, with probability $1 - \delta_1 = \delta_2$ the facility is not open, but then we

can bound the cost from above according to 11. Thus the expected cost for client j is at most $\delta_1 c_j^1 + \delta_2(c_j^1 + 2c_j^2)$.

13. Adding up the expected costs for all js (using that $\delta_2 < \delta_1$):

$$\sum_{j \in C} (\delta_1 c_j^1 + \delta_2(c_j^1 + 2c_j^2)) = \delta_1 cost(S_1) + \delta_2 cost(S_1) + 2\delta_2 cost(S_2)$$

$$\leq 2\delta_1 cost(S_1) + 2\delta_2 cost(S_2) = 2(\delta_1 cost(S_1) + \delta_2 cost(S_2))$$
$$\leq 2(3 + \delta_1 \epsilon)\mathrm{OPT} \leq 2(3 + \epsilon)\mathrm{OPT},$$

where we also used 7.