

# 7.13

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**The Problem:** Let  $G = (V, E)$  be a directed graph with source  $s$  and sink  $t$ . Every edge has infinite capacity. Every node  $v \in V$  has capacity  $c_v$  (note that the nodes  $s$  and  $t$  are included in  $V$ ). Find the maximum flow  $f$  on  $G$  subject to the constraint

$$f_{in}(v) = \sum_{e \text{ into } v} f(e) \leq c_v$$

with the other standard capacity and conservation constraints. Further, define  $s$ - $t$  cuts and their corresponding capacities on  $G$  such that the minimum capacity of all  $s$ - $t$  cuts is equal to the max flow.

**Solution:** First, consider the following construction of a graph  $G' = (V', E')$ : for every node  $v \in V$  we will create two nodes  $v_a$  and  $v_b$  such that  $V' = \{v_a : v \in V\} \cup \{v_b : v \in V\}$ . Then we will recreate all edges from  $E$  and create an edge for each pair  $v_a$  and  $v_b$  such that  $E' = \{(u_b, v_a) : (u, v) \in E\} \cup \{(v_a, v_b) : v \in V\}$ . For every edge of the form  $e = (u_b, v_a)$  we will set the capacity  $c_e = \infty$  and for every edge  $e = (v_a, v_b)$  we will set the capacity  $c_e = c_v$ . We will consider  $s_a$  to be the source and  $t_b$  to be the sink of this graph.

**Lemma 1.** The maximum flow value on  $G$  is equal to the maximum flow value on  $G'$ .

*Proof.* First, we will show that any flow  $f$  on  $G$  induces a flow  $f'$  on  $G'$  of the same value. For every edge of the form  $e' = (v_a, v_b)$  we will set  $f'(e') = f_{in}(v)$ . Note that since  $f$  is a valid flow on  $G$  we know that  $f_{in}(v) \leq c_v$ , so our capacity constraint will be satisfied on all edges of this form. Then for any edge  $e' = (u_b, v_a)$  we will set  $f'(e') = f(e)$  where  $e = (u, v)$ . Since the capacities of these edges are infinite, our capacity constraint is again satisfied. Now for every node  $w \in V'$  we must show that the conservation constraint is satisfied, namely that  $f'_{in}(w) = f'_{out}(w)$ . Note that  $f'_{in}(v_a) = \sum_{e \text{ into } v_a} f'(e) = \sum_{e \text{ into } v} f(e) = f_{in}(v)$  for any node  $v_a$ . There is only one edge out of  $v_a$ , namely  $e' = (v_a, v_b)$ , so  $f'_{out}(v_a) = f'(e')$  which is  $f_{in}(v_a)$  by construction. Hence  $f'_{in}(v_a) = f'_{out}(v_a)$ . Now for any node  $v_b$  we have that  $f'_{out}(v_b) = \sum_{e \text{ out of } v_b} f'(e) = \sum_{e \text{ out of } v} f(e) = f_{out}(v)$ . And as we just pointed out, the only edge coming into  $v_b$  is the edge  $e' = (v_a, v_b)$  which has a flow  $f'(e') = f_{in}(v)$ . Since  $f$  is a valid flow on  $G$ , it must satisfy the conservation constraint on  $G$ , therefore  $f_{in}(v) = f_{out}(v)$ . Hence it follows that  $f'_{in}(v_b) = f'_{out}(v_b)$ . Finally, we've shown that  $f'_{in}(t_b) = f'_{out}(t_a) = f'_{in}(t_a) = f_{in}(t)$ , so the flows  $f$  and  $f'$  have the same value. Note that this shows that the maximum flow value on  $G'$  must be at least as big as the maximum flow value on  $G$ .

Now we will show that any flow  $f'$  on  $G'$  induces a flow  $f$  on  $G$  of the same value. Specifically, for any edge  $e = (u, v) \in E$ , we will set  $f(e) = f'(e')$  where  $e' = (u_b, v_a) \in E'$ . Edge capacities in  $G$  are all infinite, so we will not be violating any edge capacity constraints. For any node  $v \in V$ , we see that  $f_{in}(v) = \sum_{e \text{ into } v} f(e) = \sum_{e \text{ into } v_a} f'(e)$ . Since  $f'$  is a valid flow on  $G'$ , we know that  $f'_{in}(v_a) = f'_{out}(v_a)$ , so  $\sum_{e \text{ into } v_a} f'(e) = \sum_{e \text{ out of } v_a}$ . But since

$e' = (v_a, v_b)$  is the only edge out of  $v_a$ , we have that  $f_{in}(v) = f'(e')$  where  $f'(e') \leq c_v$  by the capacity constraints on  $G'$ . Thus  $f_{in}(v) \leq c_v$ , satisfying all of our capacity constraints on  $G$ . Note that we have also know that  $f'(e) = f'_{in}(v_b) = f'_{out}(v_b)$  since  $f'$  is a valid flow on  $G'$ , and since  $f'_{out}(v_b) = f_{out}(v)$  by construction, we have that  $f_{in}(v) = f_{out}(v)$ , satisfying our conservation constraint. Finally, we know again by construction that  $f_{in}(t) = f'_{in}(t_b)$ , so the flows  $f$  and  $f'$  have the same value. This shows that the maximum flow value on  $G$  must be at least as big as the maximum value on  $G'$ , and combined with the first section of our proof we have our desired result.  $\square$

Note that the actual construction of  $G'$  is in polytime since  $|V'| = 2|V|$  and  $|E'| = |E| + |V|$ , and we can find the maximum flow on  $G'$  via the Ford-Fulkerson algorithm in polytime on  $|V'|$ , hence our algorithm for finding the maxflow on  $G$  is polytime.

Now we must define  $s$ - $t$  cuts in the context of node-capacitated graphs. We will define an  $s$ - $t$  cut  $C$  as a pair  $C = (A, B)$  where  $A \cup B = V$  and  $A \cap B = \emptyset$  with  $s \in A$  and  $t \in B$ . The capacity of such a cut will be defined as the sum of all the nodes in  $A$ , specifically  $c(C) = \sum_{v \in A} c_v$ .

**Lemma 2.** The minimum  $s$ - $t$  cut capacity on  $G$  is equal to the minimum  $s$ - $t$  cut capacity on  $G'$ .

*Proof.* First, we will show that any cut (understood to be an  $s$ - $t$  cut henceforth)  $C = (A, B)$  on  $G$  induces a cut on  $G'$  of the same capacity. To see this, consider the cut  $C' = (A', B')$  where  $A' = \{v_a : v \in A\}$  and  $B' = V' \setminus A'$ . By construction, the only edges being 'cut' in  $G'$  will be edges of the form  $(v_a, v_b)$  since  $A'$  contains no nodes of the form  $v_b$ . Thus, letting  $e_v = (v_a, v_b)$ , the capacity of the cut in  $G'$  will be  $c(C') = \sum_{e \text{ out of } A'} c_e = \sum_{v \in A} c_{e_v} = \sum_{v \in A} c_v = c(C)$ . This shows that the minimum cut capacity of  $G'$  is less than or equal the minimum cut capacity of  $G$ .

Now we will show that any finite capacity cut  $C' = (A', B')$  on  $G'$  induces a cut on  $G$  of the same capacity. We specify that we are restricting to finite capacity cuts due to the necessary existence of infinite capacity cuts in  $G'$ : for example, consider the cut where  $A' = \{s_a, s_b\}$ . Despite the existence of infinite capacity cuts, we know there is at least one finite capacity cut in  $G'$ , namely the cut where  $A' = \{s_a\}$ . This implies that the minimum cut capacity on  $G'$  is finite, so we will simply ignore any cuts of infinite capacity. Since all edges of the form  $(u_b, v_a)$  are of infinite capacity, this implies that  $A'_c = \{u \in A' : \text{there exists an edge } (u, v) \text{ such that } v \in B'\} \subseteq \{v_a : v \in V\}$ . Thus we construct the cut  $C = (A, B)$  on  $G$  where  $A = \{v : v_a \in A'_c\}$  since  $c(C) = \sum_{v \in A} c_v = \sum_{v \in A} c_{e_v} = \sum_{e \text{ out of } A'} c_e = c(C')$ . Thus we have shown that for any cut on  $G'$  we can create a cut of equal capacity on  $G$ , which in combination with the first part of this proof shows that the minimum cut capacities of  $G$  and  $G'$  are equal.  $\square$

We are now ready for our final claim:

**Theorem 3.** The maximum flow value on  $G$  is equal the minimum cut capacity on  $G$ .

*Proof.* We have shown that the maximum flow value on  $G$  is equal to the maximum flow value on  $G'$  and that the minimum  $s$ - $t$  cut capacity on  $G$  is equal to the minimum  $s$ - $t$  cut

capacity on  $G'$ . Since  $G'$  is an instance of the standard max flow problem, we know that the minimum cut capacity on  $G'$  is equal to the maximum flow value on  $G'$ . By the transitive property, this proves that the maximum flow value on  $G$  equals the minimum cut capacity on  $G$ .  $\square$