Foundations of Computer Science (FoCS)
Final Exam - Solutions

Name: ____________________

RIN: ____________________

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Grade									

Table 1: **This table is for TA use only; do not modify.**

Instructions:

- You have **2 hours** to complete this test. The test is worth a total of **100 points** and has **14 pages**, including four **scratch pages** at the end.
- You **are not allowed** to use laptops, cell phones, calculators or any other type of electronic device.
- You can have expressions of the form $\binom{m}{n}$, m^n, $m!$, etc. in your final answers.
- Cheating in a test will result to an **immediate F** for the whole course.
- **Write your answers clearly and completely!** Incomplete answers will receive very little or no credit.
- Please read each question carefully several times before beginning to work and especially before asking questions. We generally will not answer questions except when there is a glaring mistake or ambiguity in the statement of a question.

1. (**35 = 5 × 7 points**) For each of the following counting problems, state the answer in the space below.

   (a) You have 20 pennies, 30 nickels, and 40 dimes. Assume that the pennies are identical, the nickels are identical, and the dimes are identical. In how many ways can you put all the coins in a row?

   Solution: We have in total $20 + 30 + 40 = 90$ coins. Each set of coins consists of identical coins thus:
$$\#\text{ ways} = \frac{90!}{20! \cdot 30! \cdot 40!}.$$

   (b) Find the number of solutions to $x+y+z = 29$, where x, y, and z are nonnegative integers.

   Solution: We select 29 items from a set of 3 elements:
$$\#\text{ solutions} = C(3 + 29 - 1, 29) = \frac{31!}{29! \cdot (31 - 29)!} = 465.$$

   (c) Find the number of solutions to $x + y + z = 29$, where $x \geq 7$, $y \geq 7$, and $z \geq 0$ are integers.

   Solution: We select 7 items from the types x,y respectively, thus we have to select $29 - 14 = 15$ items from a set of 3 elements:
$$\#\text{ solutions} = C(3 + 15 - 1, 15) = \frac{17!}{15! \cdot (17 - 15)!} = 136.$$

   (d) Find the number of subsets of $S = \{1, 2, 3, \ldots, 10\}$ that contain exactly five elements, all of them even.

   Solution: There is only 1 way to have a set of 5 elements, that all of them are even. The reason is because S has only 5 even elements.

   (e) How many ways are there to arrange the letters of the word NONSENSE that start with the letter O?

   Solution: The first letter will be O. The remaining letters are 3 Ns, 2 Ss, and 2 Es. Assume we first arrange the Ns then the Ss and last the Es:
$$\#\text{ ways} = C(7, 3) \cdot C(4, 2) \cdot C(2, 2) = \binom{7}{3}\binom{4}{2}\binom{2}{2} = 210.$$

2. (**7 points**) Find the coefficient of x^5 in $(2 + x^2)^{12}$.

   Solution: The polynomial that is produced by this product form, has only even powers, thus the coefficient of $x^5 = 0$.

3. (**7 points**) A computer is programmed to print subsets of $\{1, 2, 3, 4, 5\}$ at random. If the computer prints 40 subsets, prove that some subset must have been printed at least twice.

   Solution: The computer can create $2^5 = 32$ different subsets. If the computer prints 40 subsets then through the Pigeonhole Principle we derive that there will be at least one subset that will be printed more than once.

4. (**7 points**) Pick a bit string (i.e., a string of zeros and ones) from the set of all bit strings of length ten. What is the probability that the bit string has the sum of its digits equal to seven?.

Solution: There are $2^{10} = 1024$ bit strings of length 10. In order the sum of the digits to be equal to 7, we have to have exactly 7 ones in the string.

$$\# \text{ ways to place 7 1s in a string of length } 10 = C(10,7) = \binom{10}{7} = 120.$$

Thus the probability the sum to be 7 is:

$$Pr\left(\sum bi = 7\right) = \frac{120}{1024} \approx 12\%.$$

5. (**15 = 5 × 3 points**) For each of the following statements determine whether they are **true** or **false**.

   (a) We can design an ambiguous context-free grammar for every context-free language.

   Solution: TRUE

   (b) The language $L = \left\{a^n b^n,\ 0 \leq n \leq 10^5\right\}$ is regular.

   Solution: TRUE

   (c) A context-free grammar for the language L is ambiguous if there exists at least one string in L that has two leftmost derivations.

   Solution: TRUE

   (d) Let $\Sigma = \{0\}$. The set of all languages over the alphabet Σ is countably infinite.

   Solution: FALSE

   (e) If $P = NP$ then the set of recursive languages is equal to the set of recursively enumerable languages.

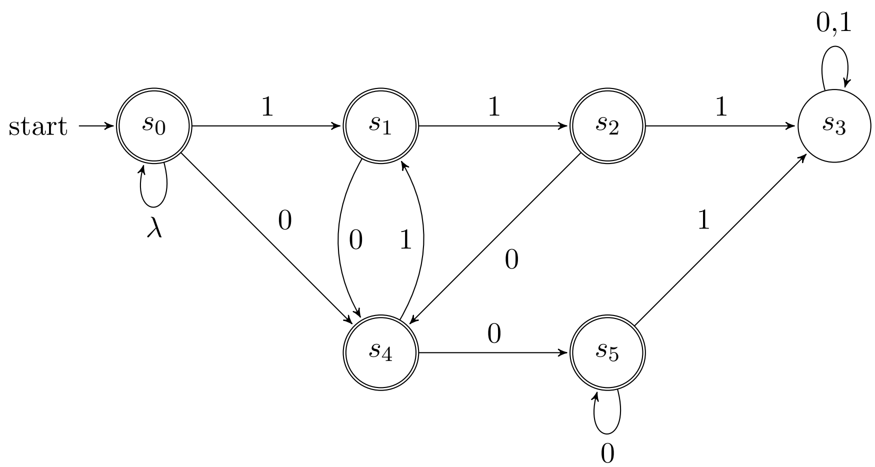
   Solution: FALSE

6. (**7 points**) Design a deterministic or non-deterministic finite automaton for the language

$$L = \{w \in \Sigma^* : \text{strings in } L \text{ do not contain 001 or 111}\},$$

where $\Sigma = \{0, 1\}$.

Solution: We can create the following deterministic FA:



7. (**6 points**) Consider the following context-free grammar over the alphabet $\Sigma = \{a, b\}$:

$$\begin{aligned} S &\rightarrow abSB \mid \lambda \\ B &\rightarrow \lambda \end{aligned}$$

Which language does it generate?

Solution: The language generated by this context-free grammar is :

$$L = \lambda, ab, abab, ababab, \ldots \text{ or } L = (ab)^*.$$

8. (**6 points**) Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the set of natural numbers. Let $S = \mathbb{N} \times \mathbb{N}$. Is S countably or uncountably infinite? Prove your answer.

Solution: The Cartesian product consists of order pairs of the form (i, j) where $i \in \mathbb{N}$ and $j \in \mathbb{N}$. We are going to define the following counting procedure, that counts all ordered pairs in $\mathbb{N} \times \mathbb{N}$:

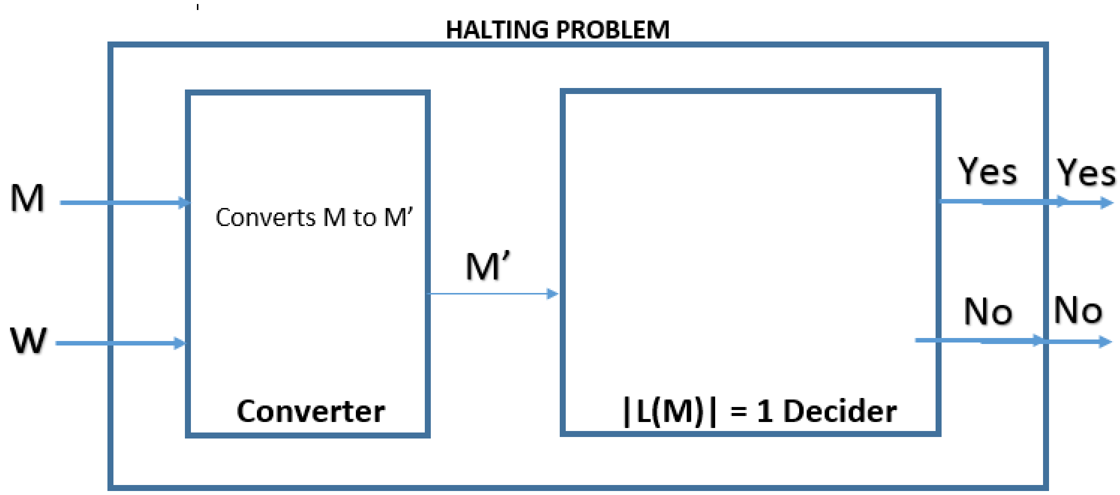
$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[dl] & (0,2) \arrow[dl] & \cdots \arrow[dl] \\
(1,0) \arrow[urr] & (1,1) \arrow[dl] & (1,2) \arrow[dl] & \cdots \arrow[dl] \\
(2,0) \arrow[uurrr] & (2,1) & (2,2) & \cdots \\
\cdots & & &
\end{tikzcd}$$

Thus the set S is <u>countably infinite</u>.

9. (**10 points**) Prove (**without using Rice's theorem**) that the following problem is undecidable:

Let $\Sigma = \{a, b\}$. Let M be a Turing Machine that accepts the language $L(M)$ over the alphabet Σ. Is $|L(M)| = 1$ (i.e., is the number of strings accepted by the Turing Machine M equal to one)?

Solution: We are going to reduce the halting problem into the $|L(M)| = 1$ problem. Assume that $|L(M)| = 1$ is decidable, which means that there exists a Turing Machine with input M that replies YES if $|L(M)| = 1$ or NO otherwise. We create the following framework:



We convert M into the Turing Machine M' that is taking s as an input string:

(a) Run M on w:
   - If M halts on w then if $w = s$ accept s and $L(M') = \{s\}$
   - If M doesn't halt on w then it gets in an infinite loop and rejects $L(M') = \{\}$

(b) If $L(M') = \{s\}$ then $|L(M)| = 1$
   - Halting Problem: Yes
   - $|L(M)| = 1$: Yes

   If $L(M') = \{\}$ then $|L(M)| = 0$
   - Halting Problem: No
   - $|L(M)| = 1$: No

So we reduced the halting problem to the $|L(M)| = 1$ problem. The halting problem is undecidable, thus the $|L(M)| = 1$ problem is undecidable as well.

Scratch page 1

Scratch page 2

Scratch page 3

Scratch page 4