and $s^{(y)}$ give rise to the same section of $\tilde{\mathcal{F}}$ over $N'_x \cap N'_y$. But, thanks to the injectivity of the natural map $\mathcal{F}(N'_x \cap N'_y) \to \tilde{\mathcal{F}}(N'_x \cap N'_y)$ we conclude that the restrictions of $s^{(x)}$ and $s^{(y)}$ to $N'_x \cap N'_y$ are the same. Since $\mathcal{F}$ is actually a sheaf, this implies that there exists $s \in \mathcal{F}(U)$ whose restriction to N'_x is $s^{(x)}$ for all $x \in U$. Thus we have $s_x = \sigma(x)$ for all $x \in U$, or, what is the same, $\tilde{s} = \sigma$, proving that $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ is surjective.

Somewhat subtler is the relationship between Axiom $\mathbf{S}_2$ and the surjectivity of $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$. If $\mathcal{F}$ satisfies $\mathbf{S}_2$, then any section of $\tilde{\mathcal{F}}$ gives rise, as above, to an open covering $\{N_x\}$ and elements $s^{(x)}$ of $\mathcal{F}(N_x)$. In order to piece all these elements together and obtain an element of $\mathcal{F}(U)$ we need to check that the restrictions of $s^{(x)}$ and $s^{(y)}$ to $N_x \cap N_y$ coincide, at least after passing to a smaller covering. We have the following set-topological lemma.

1.13. Lemma. *Let $\{U_i\}_{i \in I}$ be a locally finite open covering of a topological space U and $\{V_i\}_{i \in I}$ be a shrinking. Then for every $x \in U$, there exists an open neighbourhood M_x such that $I_x = \{i \in I : M_x \cap V_i \neq \emptyset\}$ is finite, and if $i \in I_x$ then x belongs to $\overline{V}_i$ and M_x is a subset of U_i. If M_x and M_y intersect, then there exists $i \in I$ such that $M_x \cup M_y \subset U_i$.*

Proof. Since $\{U_i\}$ is locally finite, so is the shrinking, and the existence of M_x such that the corresponding I_x is finite, is trivial. We will now cut down this neighbourhood further in order to satisfy the other conditions. We intersect M_x with $U \setminus \overline{V}_i$ for all $i \in I_x$ for which $x \notin \overline{V}_i$. We thus obtain an open neighbourhood of x, and the closures of all V_i, $i \in I_x$ then contain x. It can be further intersected with $\bigcap_{i \in I_x} U_i$, and the resulting neighbourhood satisfies the first assertion of the lemma. Now if $M_x \cap M_y \neq \emptyset$, then for any $z \in M_x \cap M_y$, choose $i \in I$ such that $z \in V_i$. Then M_x intersects V_i and hence $M_x \subset U_i$. Similarly M_y is also contained in U_i proving the second assertion.

This lemma can be used to deduce that under a mild topological hypothesis, the natural maps $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ are surjective for all open sets U, if the presheaf $\mathcal{F}$ satisfies Axiom $\mathbf{S}_2$.

1.14. Proposition. *If every open subset U of X is paracompact, and the presheaf $\mathcal{F}$ satisfies $\mathbf{S}_2$, then the map $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ is surjective for all U.*

Proof. Firstly, given an element σ of $\tilde{\mathcal{F}}(U)$, there is a locally finite open covering $\{U_i\}$ of U, and elements $s_i \in \mathcal{F}(U_i)$ for all i, with the property that for all $x \in U_i$, the elements s_i have the image $\sigma(x)$ in $\tilde{\mathcal{F}}_x$. Let $\{V_i\}$ be a shrinking of $\{U_i\}$. For every $x \in U$, choose M_x as in Lemma 1.13. We may also assume that the restrictions to M_x of any of the s_i for which $i \in I_x$, is the same, say $s^{(x)}$. It follows that the restrictions of $s^{(x)}$ and $s^{(y)}$ to $M_x \cap M_y$

are the same as the direct restriction of some s_j to $M_x \cap M_y$, proving in view of Axiom $\mathbf{S}_2$, that there exists $s \in \mathcal{F}(U)$ whose restriction to M_x is $s^{(x)}$ for all $x \in U$. This proves that $\mathcal{F}(U) \longrightarrow \widetilde{\mathcal{F}}(U)$ is surjective.

1.15. Exercises.

1) Let X be a topological space which is the disjoint union of two proper open sets U_1 and U_2. Define $\mathcal{F}(U)$ to be (0) whenever U is an open subset of either U_1 or U_2. For all other open sets U, define $\mathcal{F}(U) = A$, where A is a nontrivial abelian group. If $U \subset V$ and $\mathcal{F}(U) = A$, then define the restriction map to be the identity homomorphism. All other restriction maps are zero. Show that $\mathcal{F}$ is a presheaf such that $\tilde{\mathcal{F}} = (0)$.
2) In the above, does $\mathcal{F}$ satisfy Axiom $\mathbf{S}_2$?

Subsheaves.

1.16. Definition. A sheaf $\mathcal{G}$ is said to be *a subsheaf* of a sheaf $\mathcal{F}$ if we are given a homomorphism $\mathcal{G} \longrightarrow \mathcal{F}$ satisfying either of the following equivalent conditions.

1) $\mathcal{G}_x \longrightarrow \mathcal{F}_x$ is injective for all $x \in X$.
2) For any open subset U of X, $\mathcal{G}(U) \longrightarrow \mathcal{F}(U)$ is injective.

To see that the above conditions are equivalent, note that 1) implies that $E(\mathcal{G}) \longrightarrow E(\mathcal{F})$ is injective and hence the set of sections of $\mathcal{G}$ over any set is also mapped injectively into the set of sections of $\mathcal{F}$. Conversely, assume 2), and let $a, b \in \mathcal{G}_x$ have the same image in $\mathcal{F}_x$. Then there exist a neighbourhood U of x and elements $s, t \in \mathcal{G}(U)$ such that $s_x = a, t_x = b$. Moreover, by replacing U with a smaller neighbourhood we may also assume that the images of s and t are the same in $\mathcal{F}(U)$. This implies by our assumption that $s = t$ as elements of $\mathcal{G}(U)$, as well. Hence $s_x = t_x$ in $\mathcal{G}_x$.

2. Basic Constructions

When $\mathcal{F}$ is a sheaf, it is legitimate to call elements of $\mathcal{F}(U)$ *sections* of $\mathcal{F}$ over an open set U, since they can be identified with sections of the associated étale space. Continuous sections of the étale space make sense, on the other hand, over any subspace of X.

2.1. Proposition. *If K is a closed subspace of a paracompact topological space X, then any section over K of a sheaf $\mathcal{F}$ on X is the restriction to K of a section of $\mathcal{F}$ over a neighbourhood of K.*