ORDINARY DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

DANA D. CLAHANE

1. Introduction to Abstract Mathematics

Preview

In this chapter, we prepare the reader for the later sections in the book by introducing basic logic terminology, and modern function notation. All mathematical statements can be written as a combination of these logical terms, so in order to progress into advanced mathematics smoothly and in order to minimize unknowing errors in thinking, any serious student of mathematics should, as early as possible, begin to use these terms in problem-solving and in mathematical communication. The use of this terminology makes mathematics less intimidating, even though the logic symbols at first may look difficult. In reality, it only takes a few days of practice to get used to expressing mathematical reasoning this way, so don't be intimidated by it. As this course progresses, you will learn to appreciate why this section starts this book. The things that you learn in this section are used by mathematicians as a trick for the purpose of focusing on the important details of a mathematical statement. Using these terms when working on especially difficult parts of a problem or proof, will give you the ability to verbalize clearly what you are trying to do in the problem. You are strongly encouraged to adopt this terminology now. Use the logic you learn in this section as a foundation for the rest of the sections in the book.

1.1. **Sets. Concepts Emphasized: Set,$\{\ \}$,$\mathbb{R}$, $\mathbb{N}$,$\mathbb{Z}$, $\mathbb{Q}$, $\in$**

Definition 1.1. A set is a collection of objects, which we usually represent with symbols. Sets can contain not only numbers but other objects, or no objects or symbols at all. Often we use curly brackets $\{\}$ to enclose the objects that are in the set.

Example 1.2. $\{1, 2\}$ contains only the number 1 and the number 2.

We now look at important sets that will be used throughout the book:

Definition 1.3. $\mathbb{N}$ denotes the set of all natural or counting numbers; i.e.,

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Definition 1.4. $\mathbb{W}$ denotes the set of all whole numbers; i.e.,

$$\mathbb{W} = \{0, 1, 2, 3, \ldots\}.$$

The author would like to thank his former student Craig Luis, who performed LaTeX typsetting of an earlier version of a portion of these lecture notes. The author would also like to thank his current and former Fullerton College Linear Algebra/Differential Equations students, who spotted misprints on many occasions or pointed out mathematical errors in earlier versions. Finally, the author would like to than Professor Michel Lapidus for helpful comments on these notes.

Definition 1.5. $\mathbb{Z}$ denotes the set of all integers; i.e.,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Definition 1.6. $\mathbb{R}$ denotes the set of all real numbers. The real numbers include the integers, fractions of integers (e.g. $\frac{1}{3}$) and any number that can be written as a non-terminating, non-repeating decimal, e.g. 0.12345678910....

Definition 1.7. $\mathbb{Q}$ denotes the set of all rational numbers, including, for example, $-5/3$.

Definition 1.8. $\mathbb{C}$ denotes the set of all complex numbers, including, for example, $3+4i$, where i here denotes $\sqrt{-1}$.

Definition 1.9. $\in$ means "is an element of".

Example 1.10. 1 is an natural number, an integer and a real number. Therefore, we can write: $1 \in \mathbb{N}$, $1 \in \mathbb{Z}$ and $1 \in \mathbb{R}$.

Section Review

Symbol	What symbol represents	Examples
$\mathbb{N}$	Natural numbers	1,2,3,...
$\mathbb{W}$	Whole numbers	0,1,2,3,...
$\mathbb{Z}$	Integers	...,-2,-1,0,1,2,...
$\mathbb{R}$	Real numbers	$1, \frac{1}{2}, \pi, \sqrt{2}$
$\mathbb{C}$	Complex numbers	$3-8i$, $-5+7i$

Exercises

(1) Decide whether the following statements are true or false.
   (a) $-1 \in \mathbb{Z}$.
   (b) $-1 \in \mathbb{N}$.
   (c) $\frac{1}{2} \in \mathbb{Z}$.
   (d) $\sqrt{2} \in \mathbb{Z}$.
   (e) $\sqrt{2} \in \mathbb{R}$.

1.2. **More on Sets. Concepts Emphasized: Open interval, closed interval, R^n, difference between R^2 and open interval, $\subset$, $\cup$, $\cap$**

In general, the expression $\{x|\ P\}$, where P is a mathematical statement, to mean the set of all x such that a given statement P is true. If A is a set, then $\{x \in A|\ \mathrm{P}\}$ means the set of all x in A such that P is true.

Example 1.11. $\{a \in \mathbb{R}|\ a^2 = 1\} = \{\pm 1\}$

Example 1.12. $\{a \in \mathbb{N}|\ a^2 = 1\} = \{1\}$

Definition 1.13. $[a, b]$ is the collection of all real numbers between a and b including a and b, i.e. $[a, b] = \{c \in \mathbb{R}|\ a \leq c \leq b\}$. $[a, b]$ is called a "closed interval."

Definition 1.14. (a, b) is the collection of all real numbers between a and b not including a or b, i.e. $(a, b) = \{c \in \mathbb{R}|\ a < c < b\}$. (a, b) is called an "open interval."

Example 1.15. The closed interval $[0,1]$ contains only two numbers that the open interval $(0,1)$ doesn't: 0 and 1.

Definition 1.16.

$$\mathbb{R}^n = \{(\omega_1, \omega_2, \ldots, \omega_n) : \omega_i \in \mathbb{R} \quad \text{for all } i = 1, 2 \ldots, n\}.$$

Example 1.17. Let $\omega = (1, 1.1, 1.2, \ldots, 2)$. Then $\omega \in \mathbb{R}^{11}$, and $\omega_2 = 1.1$.

Section Review

Symbol	What symbol represents	Examples
$\mathbb{Z}^2$	$(a,b), a,b \in \mathbb{Z}$	$(1,2)$
$\mathbb{R}^2$	$(a,b), a,b \in \mathbb{R}$	$(1,\sqrt{2})$
$\mathbb{R}^4$	$(a,b,c,d), a,b,c,d \in \mathbb{R}$	$(1,\sqrt{2},0,-1)$

Exercises

(1) Decide whether the following statements are true or false.
   (a) $1 \in [0,2]$.
   (b) $1 \in \{0,2\}$.
   (c) $1 \in (0,1)$.
   (d) $1 \in (0,1]$.
   (e) $1 \in \{0,1\}$.
(2) Decide whether the following statements are true or false.
   (a) $(1,2,2) \in \mathbb{Z}^2$.
   (b) $(1,2,3) \in \mathbb{Z}^3$.
   (c) $(1,2,3) \in \mathbb{R}^3$.
   (d) $(\sqrt{3},\pi,1) \in \mathbb{R}^3$.

1.3. **Functions. Concepts Emphasized: Function, Domain, Range**

The following notation is the most important tool in a mathematician's toolbox, so you are strongly encouraged to always use it anytime you deal with functions. This notation is crucial for work beyond calculus, especially, but it will be useful in this course as well.

Definition 1.18. Recall that a function f is a relation between two sets, the first of which is called the domain of inputs and a second set, called the codomain of outputs of f. We write $f : D \to R$ precisely when we mean that f is a function with domain D and codomain R.

Section Review

Symbol	What symbol represents	Examples
$f : X \to Y$	f is a function from X to Y	$f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x + 2$ $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x + 2$

Exercises

(1) Decide whether the following statements are true or false. If you answer "false," give a reason for your conclusion.
   (a) We can define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x + 1$.
   (b) We can define $f : \mathbb{Z} \to \mathbb{R}$ by $f(x) = x + 1$.
   (c) We can define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = \frac{x}{2}$.

(d) We can define $f : \mathbb{Z} \to \mathbb{R}$ by $f(x) = \frac{x}{2}$.
(e) We can define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/x$.
(f) We can define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{x}$.
(g) We can define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$.
(h) We can define $f : (-\infty, 0] \to \mathbb{C}$ by $f(x) = \sqrt{x}$.
(i) We can define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{|x|}$.

1.4. Logical Statements. Concepts Emphasized: iff, $\exists$, $\forall$

Definition 1.19. "iff" means "if and only if".

Definition 1.20. $\exists$ means "there exists".

Example 1.21. $\exists$ at least one real solution to the equation $x + 5 = 7$.

Definition 1.22. $\forall$ means "for all".

Example 1.23. $\forall x \in \mathbb{R}$, there exists a $y \in \mathbb{R}$ such that $x + y = 0$.

Section Review

Symbol	What symbol represents	Examples
$\exists$	"There exists"	$\exists x$ such that $x + 1 = 0$ (x=-1)
$\forall$	"For all"	$\forall x \in \mathbb{N}$, $x > 0$
iff	"if and only if"	$x + 1 > 0$ iff $x > -1$

Exercises

(1) Rewrite the following statements using the symbols $\forall$, $\exists, \in, \mathbb{R}, \mathbb{Z}, \mathbb{N}$ and iff:
  (a) There exists a real number x such that $x + 1 = 0$.
  (b) For all real numbers x, $0x = 0$.
  (c) For all real numbers x, there exists y such that $x * y = 0$.
  (d) $xy = 0$ for all real numbers x if and only if $y = 0$.
  (e) $x + 1$ is an integer if and only if x is an integer.
  (f) There exists a function f from the integers to the integers such that $f(x) = 2x$.
  (g) There exists a function f from the real numbers to the real numbers such that $f(x) = x^3$.

(2) Decide whether the following statements are true or false. If you answer "false," give a reason for your conclusion.
  (a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x + y = 2$.
  (b) $\forall x \in \mathbb{N}$, $\exists y \in \mathbb{N}$ such that $y^2 = x$.
  (c) $\forall x \in \mathbb{N}$, $\exists y \in \mathbb{R}$ such that $y^2 = x$.
  (d) $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{N}$ such that $y^2 = x$.
  (e) $\exists x \in \mathbb{Z}$ such that $\forall y \in R$, $xy = 0$.
  (f) $xy \in \mathbb{Z}$ iff $x \in \mathbb{Z}$ and $x \in \mathbb{Z}$.
  (g) $xy \in \mathbb{Z}$ if $x \in \mathbb{Z}$ and $x \in \mathbb{Z}$.
  (h) $\exists f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 1$ for all $x \in \mathbb{R}$.
  (i) $\exists f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x + 1$ for all $x \in \mathbb{Z}$.
  (j) $\exists f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = \frac{x}{2}$ for all $x \in \mathbb{Z}$.

(k) $\exists f : \mathbb{Z} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$ for all $x \in \mathbb{Z}$.
(l) $\exists f : \mathbb{Z} \to \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, $f(x) = \sqrt{x}$.
(m) $\exists f : \mathbb{N} \to \mathbb{R}$ such that $\forall x \in \mathbb{N}$, $f(x) = \sqrt{x}$.
(n) $\exists f : \mathbb{Z} \to \mathbb{R}$ such that $\forall x \in \mathbb{Z}$, $f(x) = \sqrt{x}$.

2. Review: What is a differential equation?

Definition 2.1 (Differential equation)**.** An equation is called *differential* if and only if it contains first-order or higher-order derivatives (or, respectively, partial derivatives) of an unknown real-valued function y of an independent real variable x (or, respectively, several real variables $x_1, x_2, \ldots x_n$).

Example 2.2. $y' = 3y$ and $y'' = 4x$, where y is a real-valued function of a real variable x, are differential equations (de's).

Example 2.3. The following equation is also a differential equation, if z is a real-valued function of two real variables x and y:

$$\frac{\partial z}{\partial y} = -3xz.$$

Mathematicians prefer to distinguish differential equations where the unknown function depends on only one variable, or more than one variable:

Definition 2.4 (Ordinary differential equation)**.** A differential equation (hereafter, abbreviated as "de") in a real variable y is called *ordinary* iff y is a function of one real variable.

Example 2.5. The first two differential equations above are ordinary. (We use "ode" as an abbreviation for "ordinary differential equation."

Definition 2.6 (Partial differential equation)**.** A de is called *partial* (or a *p.d.e.* or *pde* iff the de is not ordinary.

Example 2.7. If z is a real-valued function of two real variables x and y, then

$$\partial z / \partial x = 3xyz$$

is a pde (a partial differential equation).

Remarks:

(1) In this course, we will only consider ode's. Pde's are considered in an upper-division course on such equations.

(2) Notice that the our definitions imply that ode's are NOT pde's and vice-versa is also true.

(3) The field of differential equations (including partial differential equations) is vast, and there are a large number of such equations that not enough is known about, such as the Euler equations, the Navier-Stokes Equations, and the N-body problem in celestial mechanics. Many mathematicians have devoted their lives to studying these types of equations, which have vast applications. For an introduction to the Navier-Stokes equations written

for students with minimal background, see the recent joint expository paper by the author with his former student Trevor Ta [**?**].[1]

(4) In first-year calculus, you saw what a de was, and what a solution was, and you also solved some basic de's. You also saw that de's have applications to
a) spring mechanics,
b) population growth,
c) compound and continuous interest, and
d) cooling of an object in a constant temperature medium,

to name only a few. We will only briefly mention these applications here and throughout the rest of these notes.

Definition 2.8 (Solution to a de)**.** We say that an equation is a *solution* to a de iff the equation does not contain any derivatives and also implies that the de is true. In particular, we say that a function f is a *solution* to a de in an unknown function y iff the equation $y = f(x)$ is a solution of the de.

Example 2.9. $y = e^x$ implies that $dy/dx = e^x$. Thus, $dy/dx = y$ if $y = e^x$, and therefore, by definition, the equation $y = e^x$ is a solution to $dy/dx = y$.

Exercise 1. Show that sin is a solution to $y' = \cos x$.

Exercise 2. Show that $x^2 + y^2 = 4$ is a solution to $dy/dx = -x/y$, under the assumption that $y \neq 0$. *Hint:* Recall from first semester calculus that you can use implicit differentation to show that the de holds here.

Definition 2.10 (Order of a de)**.** The *order* of a de in an unknown function y is defined to be the order of the highest-order derivative that appears in the de. De's of order n for $n \in \mathbb{N}$ are said to be of nth order.

Example 2.11. $y' = 3y$ has order 1 and is called "first-order."

Exercise 3. Write down an example of a 1) first-order pde and a 2) second-order pde.

Exercise 4. Write down an example of a second-order de and also a third-order de.

Definition 2.12 (Initial value problem)**.** An *initial value problem* (hereafter abbreviated as *IVP*) is a de or collection (called a *system*) of de's together with an equation or system of equations giving the values of the unknown function y and any of its first- and/or higher-order derivatives at any point in the domain of the function or its relevant derivatives at given values of the independent variable x.

Example 2.13. $y' = 3y$, $y(0) = 1$ is an initial value problem, since the first equation is a de and the second equation gives the value of the unknown function y at a prescribed value 0 of the independent variable x.

Example 2.14. $y'' = 3y'$, $y' = 3y$, $y(0) = 1$, $y'(0) = 1$ is also an IVP.

[1]Reference needed here.

Physical Example: An important problem in mathematical physics is the so-called n-body problem and associated problems, such as the still undiscovered proof (or disproof) of Saari's conjecture. Students are invited to google these terms for more information and if you are interested in doing a research project in this area, please contact the author, as he has several students involved in a working document that acts as a tutorial in this area. Don't worry, you won't be tested on such difficult problems in this course! But you are encouraged to look into them.

Remark: Clearly, physics is a major source of applications of IVP's. Often we want to be able to predict where an object will be and only know where it is at some point in time and what its velocity and/or acceleration are over time, by some formula. Thus IVP's will play a central role in this course.

3. Orthogonal trajectories of curves

Definition 3.1 (Third-variable curve families of functions of three variables)**.** Suppose that $F : D \to \mathbb{R}$, where $D \subset \mathbb{R}^3$. Then for each $c \in \mathbb{R}$, the equation

$$F(x, y, c) = 0$$

is satisfied by a collection of points $F_c \subset \mathbb{R}^2$, and we call the collection F_C of all F_c such that $c \in \mathbb{R}$, the *third-variable curve family induced by* F.

Example 3.2. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be given by $F(x, y, z) = x^2 + y^2 - z = 0$. Then for each $c \in \mathbb{R}$, F_c is the curve in $\mathbb{R}^2$ consisting of all points $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 - c = 0$. If $c < 0$, then F_c is the so-called empty curve; that is, F_c is the curve consisting of no points at all. F_0 is the origin, and if $c > 0$, then F_c is a circle with radius $\sqrt{c}$ and center at $(0, 0)$.

Definition 3.3 (Orthogonal trajectories of third-variable curves generated by real-valued functions of three variables)**.** Suppose that $F, G : D \to \mathbb{R}$, where $D \subset \mathbb{R}^3$. Assume that for all $c \in \mathbb{R}$, $F(x, y, c) = 0$ and $G(x, y, c) = 0$ are curves in $\mathbb{R}^2$ with well-defined tangent lines at each point in them. We say that G_C is an orthogonal family of trajectories for the family F_C iff for each $(x, y) \in F_c \cap G_c$, assuming that $c \in \mathbb{R}$, the tangent lines to the graphs of F_c and G_c are perpendicular.

Remark: Orthogonal families of trajectories have important geometric applications. For example, in the study of 2-dimensional heat flow, heat flows in a perpendicular direction to that of its isotherms (curves along which temperature remains constant).

Theorem 3.4 (Orthogonal Trajectory Theorem)**.** *Let $c \in \mathbb{R}$. Suppose that $F, G : D \to \mathbb{R}$, where $D \subset \mathbb{R}^3$. Assume that $F(x, y, c) = 0$ and $G(x, y, c) = 0$ are equations of curves in $\mathbb{R}^2$ with well-defined tangent lines at each point on these curves. Then G_C is an orthogonal family of trajectories for the family F_C iff for each $G_c \in G_C$, we have that $G(x, y, c) = 0$ is a solution to the ode given by*

$$\frac{dy}{dx} = \frac{-1}{f(x, y)},$$

where $(x, y) \in F_c \cap G_c$ and $f(x, y)$ is the slope of the tangent line to F_c at (x, y).

The proof of the above theorem immediately follows from the fact that perpendicularity of lines holds if their slopes are negative reciprocals of each other. Students should write out the details of this proof as a simple exercise.

Example 3.5. The curves G_c given by $y = cx$ for $c \neq 0$ and $y = 0$ form an orthogonal family of trajectories for F_C where F_c for each $c > 0$ is given by $F(x, y, c) = 0$ and $F(x, y, z) = x^2 + y^2 - c$ for all $(x, y, z) \in \mathbb{R}^3$. To prove this, recall that by implicit differentiation, $x^2 + y^2 - c = 0$ implies that $dy/dx = -x/y$, unless $y = 0$. Now if $y = 0$ for a point $(x, y) \in F_c$, then the tangent line to F_c is vertical and thus has infinite slope, which can be viewed as the negative reciprocal of 0, the slope of G_0. In either case $c = 0$ or $c \neq 0$, the slope of G_c is c, which is y/x for a point $(x, y) \in F_c \cap G_c$. But $y/x = -1/(-x/y)$. Thus by the Orthogonal Trajectory Theorem, G_C is an orthogonal trajectory family for F_C.

Exercise: Find the orthogonal trajectory family for $F : \mathbb{R}^3 \to \mathbb{R}$ given by $F(x, y, z) = x^2 + y - z$.

4. Linear ode's

Definition 4.1 (Linear ode)**.** An ode in an unknown function y is called *linear* iff $\exists n \in \mathbb{N}$, $f_0, f_1, f_2, \ldots f_n, g : Dom(y) \to \mathbb{R}$ such that the ode can be written in the form

$$\sum_{k=0}^{n} f_{n-k}(x) y^{(n-k)} = g(x).$$

Example 4.2. $y'' - 3x^2y' - y = x$ is a de that can be written in the above form if we let $n = 2$, and let f_2, f_1, and $f_0 : \mathbb{R} \to \mathbb{R}$ be respectively given by $f_2(x) = 1$, $f_1(x) = -3x^2$, and $f_0(x) = -1$, with $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x$. Thus we can see that by definition, this de is linear.

Exercise 5. Show that $\sin xy^{(3)} - y'/x = -5$ is a linear de if y is an unknown real-valued differentiable function of a real variable x.

Exercise 6. Show that $(y')^2 = y$ is a non-linear ode if y is a differentiable, real-valued function of a single real variable x.

Remark: It is a fact, mainly omitted from standard ODE textbooks, that it is usually non-trivial to show that a given ode is non-linear. However, since we will be considering linear ODE's for quite some time, the issue of proving non-linearity of a given ode is not relevant for us at this point. On the other hand, it would be nice if every ode student could verify that *some* ode is non-linear! Can you? The following problem is even more non-trivial than the one above.

Exercise 7. Prove that $y'' - (y')^2 - 1/y = x$ is not linear.

5. Particular and general solutions of ode's

Definition 5.1. A *particular solution* of an ode is a single equation or function that is a solution. A *general solution* to an ode is a family of inequivalent equations that can be written in the form $p(x, y, c_1, c_2, \dots c_n) = 0$ for some real parameters $c_1, c_2, \dots c_n$ that can take on any real value but that are constant with respect to x, where n is the order of the ode.

Example 5.2. $y = 10$ is a particular solution to $y' = 0$, because it is only one equation.

Exercise 8. Find another particular solution of the above ode.

Example 5.3. $y = c$ is a general solution to $y' = 0$ if we regard c as a real number that can take on any value that is constant with respect to the variable x.

Exercise 9. Find a general solution to $u''(t) = t^2$. Then find a particular solution. Here, assume that u is an unknown, twice-differentiable, real-valued function.

6. Existence and Uniqueness Theorem for Solutions of nth Order Linear IVP's with Continuous Coefficients

Theorem 6.1 (Solution Existence and Uniqueness Theorem for Solutions of nth Order Linear IVP's with Constant Coefficents)**.** *Suppose that*

$$\sum_{k=0}^{n} a_{n-k} y^{(n-k)} = f,$$

is an nth order and linear ODE in the unknown function y on a non-degenerate interval upon which a_j, $\forall j \in \{1, 2, \dots, n\}$, and f are real-valued and continuous functions on I. Then there is a unique solution y of the IVP consisting of the above ode together with the initial conditions $y^{(k)}(x_0) = y_k$ for each $k \in \{0, 1, 2, \dots, n\}$.

Example 6.2. Let's prove that $y = 4$ is the only solution of $y' = 0$, $y(0) = 4$.

Proof. Note that this IVP arises from a first order linear ode $y' - 0$ and has a_1 and f respectively the continuous functions 1 and 0. Notice that if $y = 4$, then $y' = 0$. Furthermore, $y(0)$ in this case is 4, so $y = 4$ definitely is *a* solution of the IVP. However, the above theorem guarantees that there a solution. So we could simply quote the theorem to prove that there *is* a solution this IVP. Now consider uniqueness. Notice that the IVP is a first order linear ODE with coefficent 1, which is a constant and hence continuous and with initial condition $y(0) = 4$. Since constants are continous on $\mathbb{R}$ we now have shown that this can be the only solution of the IVP, by the above theorem. □

Exercise 10. Show that $y' = 2x$, $y'(-1) = 1$ has a unique real-valued function that is a solution, using the above theorem. Then, find this solution!

Research Project: Find a proof of the Solution Existence and Uniqueness Theorem for nth-Order Homogeneous ODE's with Continuous Coefficients.

7. Separable ode's

Definition 7.1 (Separable ode)**.** An ode with unknown function y is called *separable* iff it can be written in the form

$$p(y)y' = q(x)$$

for some $p : Rng(y) \to \mathbb{R}$ and $q : Dom(y) \to \mathbb{R}$.

Example 7.2. Let's show that if $y : \mathbb{R} \to (-\infty, 0) \cup (0, \infty)$ is an unknown differentiable function, then the ode

$$y' = 3y$$

is separable. Since by assumption, the range of y does not contain 0, we can divide both sides of the de, obtaining

$$\frac{1}{y}y' = 3.$$

Letting $p : Rng(y) \to \mathbb{R}$ be given by $p(y) = 1/y$ and $q : \mathbb{R} \to \mathbb{R}$ be given by $q(x) = 3$, we see that the ode is separable, by definition.

Exercise 11. Show that

$$y' = 5 - y^2$$

is separable.

Exercise 12. Suppose that $y : \mathbb{R} \to \mathbb{R}$ is unknown and differentiable. Show that $y' = x + y$ is not separable.

Theorem 7.3 (Implicit Solution Theorem for Separable Ode's)**.** *Suppose that y is an unknown differentiable, real-valued function of a real variable, and consider*

$$p(y)y' = q(x),$$

where $p : Rng(y) \to \mathbb{R}$ and $q : Dom(y) \to \mathbb{R}$. Then

$$\int p(y)dy = \int q(x)dx$$

is a general solution.

Example 7.4. Consider $y' = 3x$. This is equivalent to $1y' = 3x$. We can let $p(y) = 1$ and $q(x) = 3x$, obtaining that $\int 1dy = \int 3xdx$ is a general solution, by the Implicit Solution Theorem for Separable Ode's. We can also verify this fact directly without use of the theorem: Assuming that this equation holds, then we have that there are constants C_1 and C_2 such that $y + C_1 = 1.5x^2 + C_2$, where C_1 and C_2 are arbitrary constants. Thus $y - 1.5x^2 + C_1 - C_1 = 0$, where C_1 and C_2 are constants. Letting $c = C_1 - C_2$, we obtain that $y - 1.5x^2 + c = 0$. Consider the equivalent equation $y = 1.5x^2 + C$. Then $y' = 3x$. That is, $\int dy = \int 3xdx$ is a general solution to $y' = 3x$.

We now prove the above theorem:

Proof. Suppose that $\int p(y)dy = \int q(x)dx$. Let $P \in \int p(y)dy$ and $Q \in \int q(x)dx$. Then $\int p(y)dy = \int q(x)dx$ implies that there are $C_1, C_2 \in \mathbb{R}$ such that for all $x \in Dom(y)$, we hav that $P(y) + C_1 = Q(x) + C_2$. Differentiating both sides of this equation with respect to x yields, by the Chain Rule, $p(y)y' = q(x)$. Thus $\int p(y)dy = \int q(x)dx$ is a solution to the ode. To see that it is a general solution, notice that this integral equation is a family of equations that can be written as a single equation involving a constant C that can be chosen arbitrarily and that does not depend on x. This completes the proof of the theorem. □

8. A procedure for solving separable ode's

The above theorem unfortunately does not give function solutions for any separable ode. However, the following is a workable procedure for finding function solutions to separable ode's:

(1) Write the ode in the form

$$p(y)\frac{dy}{dx} = q(x).$$

(2) Multiply both sides by dx, obtaining

$$p(y)dy = q(x)dx.$$

(3) Integrate both sides of the equation found in (2) above. The resulting equation is a solution of the ode.

(4) If possible, solve the resulting equation for y, as it is usually preferable to have an explicit formula for y.

Example 8.1. Suppose that $y : \mathbb{R} \to (0, \infty)$ is differentiable and does not contain 0 in its range. Consider the equation

$$y' = 3y.$$

We write it in the form

$$\frac{dy}{dx} = 3y,$$

which can also be written as

$$\frac{dy}{y} = 3dx.$$

Integrating both sides yields

$$\ln|y| + C_1 = 3x + C_2.$$

Since y only takes on positive values, we have that

$$\ln y + C_1 = 3x + C_2.$$

Letting $c = C_2 - C_1$, we see that

$$\ln y = 3x + c.$$

The reader can easily obtain that this equation is a solution of $y' = 3y$, but now we solve the above equation for y:

$$e^{\ln y} = e^{3x+c},$$

which implies that $y = e^{3x+c} = e^{3x}e^c$. Letting $K = e^c$, we see that $y = Ke^{3x}$. The reader can now check that this choice of y is a general solution of $y' = 3y$. Since c can be any constant, K can be any positive constant, so letting $K = 2$, for example, we see that $y = 2e^{3x}$ is a particular solution of $y' = 3y$.

IMPORTANT Remark: ALWAYS check that any solution you find for an ode is actually a solution. Not doing do so can lead to regret, especially on exams!

Exercise 13. Solve $dy/dx = -200y$, $y(0) = -1$.

Exercise 14. Solve $xy' = 26y$.

9. Monic First-Order Linear Ode's

The following fact immediately follows from the definition of a linear ode:

Lemma 9.1 (General form of a first-order linear ode)**.** *Any first-order linear ode in an unknown function y can be written in the form*

$$u(x)y' + v(x)y = w(x) \tag{1}$$

for some $u, v, w : Dom(y) \to \mathbb{R}$.

Example 9.2. Notice that letting $n = 1$, and also letting f_1, f_0, and $g : \mathbb{R} \to \mathbb{R}$ be given by $f_1(x) = 1$, $f_0(x) = -1$, and $g(x) = 0$ in the definition of linear ode, we see that

$$y' - y = 0$$

is linear. The above lemma guarantees that Equation (1) holds for some $u, v, w : Dom(y) \to \mathbb{R}$. (In fact, here, $u = f_1$, $v = f_2$, and $w = g$.)

Exercise 15. Show that $y' = 2x - y$, given y is an unknown, real-valued function of a real variable, can be written in the form of Equation (1).

Exercise 16. Prove the above lemma.

Definition 9.3 (Monic first-order linear ode and standard form)**.** We call a first-order linear ode in an unknown function y *monic* iff it can be written in the form

$$y' + p(x)y = q(x)$$

for some $p, q : Dom(y) \to \mathbb{R}$. The above equation is called the *standard form* of the ode.

Example 9.4. The first-order linear ode $y' - y = 0$ can be written in the form $1y' + (-1)y = 0$, so $y' - y = 0$ is a monic, first-order linear ode with $p(x) = -1$ and $q(x) = 0$. $1y' + (-1)y = 0$ is the standard form of $y' - y = 0$.

Exercise 17. Show that if $y : (0, \pi) \to \mathbb{R}$ is an unknown function, then

$$xy' - (\sin x)y = x + 3$$

is a monic first-order linear ode. Show that this first-order linear ode is NOT monic if $(0, \pi)$ above is replaced by $\mathbb{R}$. Be careful!

The following example will be central to our discussion of the method of integrating factors for solving monic first-order linear ode's:

Example 9.5. Consider the first-order DE

$$xy' = x^2 + 5y, \tag{2}$$

where $y : (0, \infty) \to \mathbb{R}$ is unknown.[2] This de is equivalent to

$$xy' - 5y = x^2.$$

Since $x \neq 0$, we can rewrite the above equation in the form (by dividing both sides by x)

$$y' - \frac{5}{x}y = x, \tag{3}$$

and we see that letting $p(x) = -5/x$ for $x \neq 0$ and letting $q(x) = x$ leads to the conclusion that the original de (2) is a first order, linear de with standard form given by Equation (3).

The name "integrating factor" below will be explained later in this section.

Theorem 9.6 (Integrating Factor Theorem for Monic, First Order, Linear DE's)**.** *Suppose that $y : I \to \mathbb{R}$ is an unknown, real-valued function, where I is an open interval in $\mathbb{R}$. Let $p, q : I \to \mathbb{R}$. Then f is a solution of*

$$y' + p(x)y = q(x)$$

iff there is a $C \in \mathbb{R}$ such that f is given by

$$f(x) = \frac{1}{v(x)}[G(x) + C],$$

where $v : I \to \mathbb{R}$ is given by

$$v(x) = e^{P(x)},$$

$P \in \int p(x)dx$, and $G \in \int v(x)q(x)dx$.

Example 9.7. Consider the de (2) given in the previous example. First, we calculate $v(x) = e^{P(x)}$, where $P \in \int(-5/x)dx = -5\ln|x| + C = -5\ln x + C$, since $x > 0$. Since P can be chosen to be any of these antiderivatives, we will choose P given by $P(x) = -5\ln x$ Therefore, we have that v is given by

$$v(x) = e^{-5\ln x} = (e^{\ln x})^{-5} = \frac{1}{x^5}.$$

Hence, vq is given by

$$v(x)q(x) = \frac{1}{x^5}x = \frac{1}{x^4} = x^{-4}.$$

It follows that an antiderivative G for vq on $(0, \infty)$ is given by

$$G(x) = \frac{x^{-3}}{-3} = \frac{-1}{3x^3}.$$

[2]Hopefully, by now, the reader realizes that by "unknown," we merely mean that we have not yet determined what y is, although of course, this de, for example, has probably been solved before by many people in the past. Also, it is hoped that the reader realizes that in any de, the independent variable is assumed to be in $\mathrm{Dom}(y)$, the domain of y.

Therefore, by the Integrating Factor Theorem for First Order, Linear De's, f is a solution of the de (2) iff there is a $C \in \mathbb{R}$ such that $\forall x > 0$,

$$f(x) = \frac{1}{v(x)}[G(x) + C] = \frac{1}{\frac{1}{x^5}}\left[\frac{-1}{3x^3} + C\right] = x^5\left[\frac{-1}{3x^3} + C\right] = \frac{-x^2}{3} + Cx^5.$$

Let's check to see that f here actually is a solution by simply substituting it into the de. Letting $y = f(x)$ for the choice of f above, we obtain that

$$\begin{aligned} xy' &= x\left(\frac{-2x}{3} + 5Cx^4\right) \\ &= \frac{-2x^2}{3} + 5Cx^5. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} x^2 + 5y &= x^2 + 5\left(\frac{-x^2}{3} + Cx^5\right) \\ &= \frac{3x^2}{3} - \frac{5x^2}{3} + 5Cx^5 \\ &= \frac{-2x^2}{3} + 5Cx^5 \\ &= xy', \end{aligned}$$

which shows that the y that we have found indeed is a solution.

Exercise 18. Solve $dy/dx = x - y$.

Exercise 19. Prove the above theorem.

v above is called the *integrating factor* for the DE (2). To be precise, we will define integrating factors for first order linear ode's below:

Definition 9.8 (Integrating factor for a monic, first-order, linear ode)**.** Suppose that $y : I \to \mathbb{R}$ is an unknown, real-valued function, where I is an open interval in $\mathbb{R}$. Let $p, q : I \to \mathbb{R}$. Then $v : I \to \mathbb{R}$ is called an *integrating factor* for a monic first-order linear ode with standard form

$$y' + p(x)y = q(x),$$

iff v is given by

$$v(x) = e^{P(x)},$$

where $P \in \int p(x)dx$.

Example 9.9. Note in the example above that we calculated an integrating factor v to be given by

$$v(x) = \frac{1}{x^5}.$$

Notice that above, we have shown by use of the Integrating Factor Theorem for First Order Linear DE's that y is a solution to the DE (2) iff y is given by $y = -2x^2/3 + Cx^5$ for some constant $C \in \mathbb{R}$. Often, theorems that characterize solutions to DE's are not easy to remember. It turns out that being familiar with a technique that can be used to prove such a theorem is more convenient than trying to memorize and apply the theorem.

Therefore, let's assume for the moment that we do not know the Integrating Factor Theorem for First Order Linear DE's, and let's solve the DE (2) without use of that Theorem: First, we multiply both sides of Equation (2) by v, obtaining

$$\frac{1}{x^5}\left(y' - \frac{5}{x}y\right) = x\frac{1}{x^5}.$$

We then simplify both sides, thus obtaining

$$\frac{1}{x^5}y' - \frac{5}{x^6}y = \frac{1}{x^4}.$$

Now $-5/x^6 = -5x^{-6} = (x^{-5})' = v'$, so we can rewrite the above equation as

$$vy' + v'y = \frac{1}{x^4}.$$

However, the left side of this equation is the derivative of vy with respect to x by the product rule, so we can rewrite the above equation as

$$\frac{d}{dx}(vy) = \frac{1}{x^4}.$$

Since these quantities are the same, their families of antiderivatives must be the same; that is, we have that

$$\int \frac{d}{dx}(vy)dx = \int \frac{1}{x^4}dx.$$

Therefore, we have that

$$vy = \int x^{-4}dx,$$

which implies that there is a $C \in \mathbb{R}$ such that $\forall x > 0$,

$$\frac{1}{x^5}y = \frac{-1}{3x^3} + C.$$

Multiplying both sides of the above equation by x^5 yields

$$y = \frac{-1}{3x^3}x^5 + Cx^5.$$

Simplifying the right hand side above, we obtain that there is a constant $C \in \mathbb{R}$ such that for all $x > 0$, $y(x)$ is given by

$$y(x) = \frac{-x^2}{3} + Cx^5,$$

which is exactly the same solution that we obtained earlier by use of the Integrating Factor Theorem for Monic, First Order Linear Ode's. Recall that we have already checked that y given above is indeed a solution.

Exercise 20. Solve $y' = x + y$, assuming that y is a real-valued, differentiable function of a single real variable x.

10. r-homogeneous functions of several real variables, r-homogeneous ode's, homogeneous ode's, and a change of variables solution theorem for homogeneous ode's

In the last section, we saw that monic, first-order linear ode's can be handled with the Integrating Factor Theorem for such ode's, all of which can be written in the form

$$y' = -p(x)y + q(x).$$

Notice that the right side is a function of both x and y. It is natural to consider whether there is a method that handles the case where the above equation's right side is replaced by *any* real-valued function f of x and y. In general, this is a difficult problem to solve; however, in the case that f is a *homogeneous function* of the variables x and y, we will see that a substitution $V = y/x$ can be used to convert

$$y' = f(x, y)$$

into a separable first-order linear ode, which we have already learned how to solve. Before we state that theorem, however, we need to define what an r-homogeneous function of several real variables is, and in particular, what it means for a function of several real variables to be homogeneous. We will also need to define what it means for an ode to be r-homogeneous, and finally, homogeneous.

Definition 10.1 (r-homogeneous functions of several real variables)**.** Let $r \geq 0$. Suppose that $D \subset \mathbb{R}^n$, and let $f : D \to \mathbb{R}$.[3] Let[4] We say that f is *r-homogeneous* iff $\forall (x_1, x_2, \ldots x_n) \in D$ and all $t > 0$ such that $(tx_1, tx_2, \ldots, tx_n) \in D$, we have that

$$f(tx_1, tx_2, \ldots tx_n) = t^r f(x_1, x_2, \ldots x_n).$$

f is called *homogeneous* iff it is 0-homogeneous.

Example 10.2. Let D be the first quadrant in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be given by

$$f(x, y) = \frac{x^2 + xy}{xy + y^2}.$$

If $t > 0$ and $(x, y) \in D$, then $(tx, ty) \in D$. Also, we have that

$$f(tx, ty) = \frac{(tx)^2 + txty}{txty + (ty)^2} = \frac{t^2x^2 + t^2xy}{t^2xy + t^2y^2} = \frac{t^2}{t^2}\frac{x^2 + xy}{xy + y^2} = \frac{x^2 + xy}{xy + y^2} = f(x, y).$$

Hence f here is 0-homogeneous, by definition. Also, by definition, f is homogeneous, since it is 0-homogeneous.

Exercise 21. Let D be the first octant in $\mathbb{R}^3$. Show that $f : D \to \mathbb{R}$ given by

$$f(x, y, z) = \frac{5x^2y + xyz + xz^2}{z^3}$$

is homogeneous.

---

[3]In this course, we are concerned with the case that $n = 2$.

[4]We are also primarily concerned here with the case that $r = 0$.

Example 10.3. Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = x^2 + x$. If $t, x > 0$, then $xt > 0$, and we have that

$$f(tx) = (tx)^2 + tx.$$

If $t = 2$, which is > 0, then

$$f(tx) = f(2x) = (2x)^2 + 2x = 4x^2 + 2x,$$

which is not equal to $f(x) = x^2 + x$ when, for example, $x = 1$. Thus, f is not 0-homogeneous.

Exercise 22. (a) Can you find a function f that is 2-homogeneous?
(b) Can you find a function f that is 0.5-homogeneous?
(a) Can you find a function f that is r-homogeneous for some $r > 0$ but not s-homogeneous for some $s > 0$?
(b) Can you find a function f that is not r-homogeneous $\forall r > 0$?

Definition 10.4 (r-homogeneous first-order ode)**.** Suppose that $r \geq 0$. An ode in an unknown function y is called *r-homogeneous* iff the ode can be written in the form

$$y' = f(x, y)$$

for some r-homogeneous, real-valued function f. The ode is called *homogeneous* iff f in the above equation is 0-homogeneous.

Example 10.5. Let D be the first quadrant in $\mathbb{R}^2$, and define $f : D \to \mathbb{R}$ by

$$f(x, y) = \frac{x - 2y}{2x + y}.$$

If $t > 0$, then tx and ty are both positive, and we then have that $(tx, ty) \in D$ if $(x, y) \in D$. Furthermore, $\forall t > 0$ and all $(x, y) \in D$, we have that

$$f(tx, ty) = \frac{tx - 2ty}{2tx + ty} = \frac{t}{t}\frac{x - 2y}{2x + y} = \frac{x - 2y}{2x + y} = f(x, y).$$

Thus f is 0-homogeneous by definition, and we can also say that f is homogeneous. Hence, if $y : D \to \mathbb{R}$ is an unknown function, then

$$y' = \frac{x - 2y}{2x + y}$$

is a 0-homogeneous ode, by definition. It follows that this ode is also, simply homogeneous, by definition.

Exercise 23. Is $y' = x^2 + x$ a homogeneous ode? Prove your answer.

Exercise 24. Is $y' = x + y$ a homogeneous ode? Prove your answer!

We now state a property of homogeneous functions of two variables that will in turn prepare us to state a change-of-variables result that can be used to transform homogeneous ode's to separable ode's, which we already know how to handle:

The following result is a clarification of Theorem 1.8.3 in the Annin/Goode text:

Theorem 10.6 (One-variable Transformation Theorem for Homogeneous Functions of Two Variables)**.** *Suppose that $D \subset \mathbb{R}^2$, and assume that $\forall(x, y) \in D$, $x \neq 0$. Let $f : D \to \mathbb{R}$. Then f is homogeneous iff $\exists E \subset \mathbb{R}$ and $F : E \to \mathbb{R}$ such that $f(x, y) = F(y/x)$ $\forall(x, y) \in D$.*

Example 10.7. Let D be the first quadrant in $\mathbb{R}^2$, and define $f : D \to \mathbb{R}$ by

$$f(x, y) = \frac{x - 2y}{2x + y}.$$

The One-Variable Transformation Theorem for Homogeneous Functions of Two Variables guarantees that since, as we showed above, f is homogeneous, there is an $E \subset \mathbb{R}$ and $F : E \to \mathbb{R}$ such that

$$F(y/x) = \frac{x - 2y}{2x + y} \ \forall(x, y) \in D.$$

To find F, let $V = y/x$, so that $y = xV$. Then replace y in the expression for $f(x, y)$ by $y = Vx$ and leave the x's unchanged. Then we have that

$$F(V) = F(y/x) = \frac{x - 2Vx}{2x + Vx} = \frac{x}{x}\frac{1 - 2V}{2 + V} = \frac{1 - 2V}{2 + V}.$$

Exercise 25. Let $D \subset \mathbb{R}^2$ and assume that D is in Quadrant I. Let $f : D \to \mathbb{R}$ be given by

$$f(x, y) = \frac{2x - 3y}{3x + 2y}.$$

Use the One-Variable Transformation Theorem for Homogeneous Functions of Two Variables to show that f is homogeneous.

Exercise 26. Prove the One-Variable Transformation Theorem for Homogeneous Functions of Two Variables.

We are now prepared to state the following theorem, which guarantees that a certain substitution (change of variables) can be used to transform a homogeneous ode into a separable ode:

Theorem 10.8 (Change of Variables Theorem for Homogeneous Ode's)**.** *Suppose that $y : D \to \mathbb{R}$ is an unknown function, and let*

$$y' = f(x, y)$$

be homogeneous. Assume that $0 \notin D$, and let $E \subset \mathbb{R}$ and $F : E \to \mathbb{R}$ be such that $f(x, y) = F(y/x)$ $\forall(x, y) \in D$. Let $V = y/x$. Then a solution to $y' = f(x, y)$ is $y = Vx$, where V is a solution to the following ode, which is separable:

$$\frac{1}{F(V) - V}V' = \frac{1}{x}.$$

Example 10.9. Consider the ode

$$y' = \frac{x - 2y}{2x + y},$$

where $x, y > 0$. Recall that we showed that the right side of this equation defines a homogeneous function, so this ode is homogeneous by definition.

We let $y = Vx$ and invoke the Change of Variables Theorem for Homogeneous Ode' to obtain that a solution is $y = Vx$, where V is a solution to the separable ode

$$\frac{1}{\frac{1-2V}{2+V} - V} V' = \frac{1}{x}.$$

The above equation can be rewritten as

$$\frac{2+V}{1-2V-V(2+V)} V' = \frac{1}{x},$$

which can be rewritten in turn as

$$\frac{2+V}{1-2V-2V-V^2} V' = \frac{1}{x}.$$

We further rewrite the above equation as

$$\frac{2+V}{1-4V-V^2} \frac{dV}{dx} = \frac{1}{x},$$

which we write further using our method of solving separable ode's, as

$$\frac{-V-2}{V^2+4V-1} dV = \frac{1}{x} dx.$$

Integrating both sides indefinitely, we obtain that

$$\int \frac{-V-2}{V^2+4V-1} dV = \int \frac{1}{x} dx,$$

which we rewrite in the form

$$\int \frac{-V-2}{V^2+4V-1} dV = \ln|x| + C$$

for some constant C that does not depend on x. Since $x > 0$, we can simplify the ode into

$$\int \frac{-V-2}{V^2+4V-1} dV = \ln x + C.$$

Using u-substitution with $u = V^2 + 4V - 1$, we see that $du = (2V+4)dV = 2(V+2)dV = -2(-V-2)dV$. Therefore, the left side of the above ode can be rewritten as

$$-\frac{1}{2} \int \frac{du}{u} = \ln x + C,$$

or

$$-\frac{1}{2} \ln|u| + C_1 = \ln x + C,$$

where C_1 and C are constants that do not depend on x. Letting $K = C - C_1$, we see that we can write the above equation as

$$-\frac{1}{2} \ln|V^2 + 4V - 1| = \ln x + K.$$

We multiply both sides of the above equation by -2, yielding that

$$\ln|V^2 + 4V - 1| = -2 \ln x - 2K.$$

Taking the natural exponential of both sides of the above equation, we obtain that

$$e^{\ln|V^2+4V-1|} = e^{-2\ln x - 2K},$$

which can be rewritten as

$$|V^2 + 4V - 1| = e^{-2\ln x}e^{-2K}.$$

Since C_1 and C, and, in turn, K can be any real number, e^{-2K} can be any positive number, which we call H for now. Thus we can rewrite the above equation as

$$V^2 + 4V - 1 = \pm He^{\ln x^{-2}},$$

which can further be rewritten as

$$V^2 + 4V - 1 = Lx^{-2},$$

where L is any non-zero constant that does not depend on x. We now solve the above quadratic equation in V using the quadratic formula, starting by rewriting the equation as

$$V^2 + 4V - 1 - Lx^{-2} = 0.$$

Hence, by the quadratic formula, we have that

$$\begin{aligned}
V &= \frac{-4 \pm \sqrt{4^2 - 4(-1 - Lx^{-2})}}{2} \\
&= \frac{-4 \pm \sqrt{4}\sqrt{4 - (-1) + \frac{L}{x^2}}}{2} \\
&= \frac{2\left(-2 \pm \sqrt{5 + \frac{L}{x^2}}\right)}{2} \\
&= -2 \pm \sqrt{5 + \frac{L}{x^2}} \\
&= \frac{-2x}{x} \pm \sqrt{\frac{5x^2 + L}{x^2}} \\
&= \frac{-2x}{x} \pm \frac{\sqrt{5x^2 + L}}{\sqrt{x^2}} \\
&= \frac{-2x}{x} \pm \frac{\sqrt{5x^2 + L}}{x}.
\end{aligned}$$

Hence, we have that $y = Vx = -2x \pm \sqrt{5x^2 + L}$, where L is any non-zero, real constant with respect to x, is a general solution for

$$y' = \frac{x - 2y}{2x + y}.$$

Exercise 27. Check that y given above is a general solution to the above ode.

Exercise 28. Solve:

$$y' = \frac{x - y}{x + y}.$$

Exercise 29. Prove the One-Variable Transformation Theorem for Homogeneous Functions of Two Variables Theorem.

11. Bernoulli Ode's and their solutions

Definition 11.1 (Bernoulli Ode)**.** *An ode in an unknown function y is called* Bernoulli *iff $\exists p, q : Dom(y) \to \mathbb{R}$ and $r \in \mathbb{R}$ such that the ode can be written in the form*

$$y' + p(x)y = q(x)y^r.$$

Example 11.2. All monic first-order linear ode's are Bernoulli, since any such ode can be written in the form $y' + p(x)y = q(x)y^0$ (here, $r = 0$). For example, $y' - y = 0$ is Bernoulli.

Example 11.3. $y' = y^2$ is Bernoulli, but not linear, assuming that y is an unknown, real-valued function of a single real variable, since it can be written as $y' + 0y = 1y^2$. Here, $p(x) = 0$, $q(x) = 1$, and $r = 2$.

Exercise 30. Prove that the above ode is non-linear. Thus not all Bernoulli ode's are linear (in fact, most are not).

Exercise 31. Assuming that y is an unknown real-valued function of a real variable, determine whether the following equations are Bernoulli ode's or not, proving your answers.
(a) $y' + xy - e^x y^3 = 0$.
(b) $x + y^2 = 7$
(c) $y'' - y' + 2y = 4$.

Theorem 11.4 (Bernoulli Ode Change of Variables Theorem)**.** *Suppose that we are given a Bernoulli ode of the form*

$$y' + p(x)y = q(x)y^r, \tag{4}$$

where $p, q : Dom(y) \to \mathbb{R}$ and $r \in \mathbb{R}$. Then
(a) The ode is linear iff either $r = 0$ or $r = 1$.
(b) If $r \neq 0, 1$, and we let $u = y^{1-r}$, then the ode

$$u' + (1 - r)p(x)u = (1 - r)q(x)$$

is a monic, first-order, linear ode in u.
(c) A solution to the above ode (4) when $r \neq 1$ is in turn given by

$$y = u^{\frac{1}{1-r}}.$$

Example 11.5. Consider the Bernoulli ode $y' = y^2$, where y is an unknown, real-valued function of a single real variable. It can be rewritten in the form $y' + 0y = y^2$. Here, $p(x) = 0$, $q(x) = 1$, and $r = 2$. Per the Bernouli Ode Change of Variables Theorem, we let $u = y^{1-2} = y^{-1} = 1/y$. Part (b) of the Theorem tells us that

$$u' + (1 - 2)0u = (1 - 2)q(x) \tag{5}$$

is a monic first-order linear ode in u. We rewrite the ode in the form

$$u' = -1.$$

Thus $u(x) = -x$ is a solution to the ode (5). By Part (c) of the Bernoulli Ode Change of Variables Thorem, one obtains that

$$y = u^{\frac{1}{1-2}} = u^{-1} = 1/u = \frac{1}{-x} = \frac{-1}{x}$$

is a particular solution to $y' = y^2$.

Exercise 32. Find a particular solution to the ode

$$y' + xy = y^2.$$

Exercise 33. Repeat the above exercise but with 2 replaced by 3.

Exercise 34. Prove the Bernoulli Ode Change of Variables Theorem.

12. Exact Ordinary Differential Equations

Lemma 12.1. *Suppose that J is an open interval in $\mathbb{R}$ and that $y : J \to \mathbb{R}$ is differentiable and unknown. Let*

$$\{(x, y(x)) : x \in J\} \subset D \subset \mathbb{R}^2.$$

Suppose that $M, N : D \to \mathbb{R}$. Then $M(x, y)dx + N(x, y)dy = 0$ is an ode.

Example 12.2. Suppose that $y : \mathbb{R} \to \mathbb{R}$ is differentiable, and, say, unknown. Let $D = \{(x, y(x)) : x \in \mathbb{R}\}$. Then $ydx + dy = 0$ is an ode, by the above lemma with $M, N : D \to \mathbb{R}$ given respectively by $M(x, y) = y$ and $N(x, y) = x$.

Exercise 35. Prove that the above ode is an ode without using the lemma.

Definition 12.3 (Exact)**.** Suppose that J is an open interval in $\mathbb{R}$ and that $y : J \to \mathbb{R}$ is differentiable and unknown. Let

$$\{(x, y(x)) : x \in J\} \subset D \subset \mathbb{R}^2.$$

Suppose that $M, N : D \to \mathbb{R}$. We say that $M(x, y)dx + N(x, y)dy = 0$ is *exact*, and call ϕ a *potential* for this ode, iff the following conditions hold:

(1) $\phi : D \to \mathbb{R}$,
(2) ϕ is partially differentiable,
(3) $\phi_x = M$, and
(4) $\phi_y = N$.

Example 12.4. Consider the last example, and let $\phi : D \to \mathbb{R}$ be given by $\phi(x, y) = xy$. Let's prove, for example, that ϕ is a potential for $ydx + xdy = 0$. By its definition, ϕ satisfies Condition (1) above, and since ϕ is a polynomial function, it is partially differentiable. That is, Condition (2) holds above. Also, we have that $\forall (x, y) \in D$,

$$\phi_x(x, y) = \frac{\partial}{\partial x}(xy) = y = M(x, y),$$

and

$$\phi_y(x, y) = \frac{\partial}{\partial y}(xy) = x = N(x, y).$$

Thus, Conditions (3) and (4) hold, and $ydx + xdy = 0$ is indeed exact, with potential ϕ, by definition.

Exercise 36. Show that $ydx - xdy = 0$ is not exact.

Theorem 12.5 (Exact ODE Potential Solution Theorem)**.** *Suppose that J is an open interval in $\mathbb{R}$ and that $y : J \to \mathbb{R}$ is differentiable and unknown. Let*

$$D = \{(x, y(x)) : x \in J\}.$$

Suppose that $M, N : D \to \mathbb{R}$ and that $M(x,y)dx + N(x,y)dy = 0$ is exact. Then an equation is a solution of this ode iff the equation has the property that $\exists c \in \mathbb{R}$ and a potential ϕ for this ode, such that the equation can be rewritten as

$$\phi(x, y) = c.$$

Example 12.6. Consider the above example. We already showed that the ode is exact and that a potential for the ode is $\phi : D \to \mathbb{R}$ given by $\phi(x, y) = xy$. The Exact ODE Potential Solution Theorem guarantees that if some equation is a solution of this ode in particular, then there must be $c \in \mathbb{R}$ such that the equation can be rewritten as $xy = c$.

Exercise 37. Find the general solution to $2dx - 1,000dy = 0$.

Research Project: Prove the Exact ODE Potential Solutions Theorem.

Theorem 12.7 (Exactness Test)**.** *Suppose that J is an open interval in $\mathbb{R}$ and that $y : J \to \mathbb{R}$ is differentiable and unknown. Let*

$$\{(x, y(x)) : x \in J\} \subset D \subset \mathbb{R}^2.$$

Suppose that $M, N : D \to \mathbb{R}$ and that D is simply connected. Then $M(x,y)dx+N(x,y)dy = 0$ is exact iff $M_y = N_x$.

Example 12.8. Suppose that $J = \mathbb{R}$ and let $y : J \to \mathbb{R}$ be an unknown differentiable function. Let $D = \mathbb{R}^2$, and let $M, N : D \to \mathbb{R}$ be given by $M(x, y) = y$ and $N(x, y) = x$, for example. Then the equation $M(x, y)dx + N(x, y)dy = 0$ is in this case $ydx + xdy = 0$. Let's use the Exactness Test to show that this equation is exact.

Indeed, we have that

$$M_y(x, y) = \frac{\partial}{\partial y} y = 1 \text{ and } N_x(x, y) = \frac{\partial}{\partial x} x = 1, \ \ \forall (x, y) \in \mathbb{R}^2.$$

Thus $M_y = N_x$ here, and $ydx + dy = 0$ is, therefore, exact, by the Exactness Test.

Exercise 38. Let $y : \mathbb{R} \to \mathbb{R}$ be an unknown and differentiable. Apply the Exactness Test to show that $ydx - xdy = 0$ is not an exact equation.

Exercise 39. Prove the Exactness Test.

Definition 12.9 (Integrating factor for $M(x, y)dx + N(x, y)dy = 0$)**.** Suppose that J is an open interval in $\mathbb{R}$ and that $y : J \to \mathbb{R}$ is differentiable and unknown. Let

$$\{(x, y(x)) : x \in J\} \subset D \subset \mathbb{R}^2.$$

Suppose that $M, N : D \to \mathbb{R}$. We say that I is an *integrating factor* of $M(x, y)dx + N(x, y)dy = 0$ iff the following conditions hold:

(1) $I : D \to \mathbb{R}$,

(2) $I(x,y) \neq 0 \ \forall (x,y) \in D$, and
(2) we have that

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

is exact.

Example 12.10. Suppose that $y : \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$ is unknown and differentiable. Consider the ode

$$\frac{y}{x}dx + 1dy = 0.$$

Let $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. Let's show that $I : D \to \mathbb{R}$ given by $I(x,y) = x$ is an integrating factor for this ode.

Here, let $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. Here, $M, N : D \to \mathbb{R}$ are given by

$$M(x,y) = \frac{y}{x} \text{ and } N(x,y) = 1,$$

Now let $\phi : D \to \mathbb{R}$ be given by $\phi(x) = xy$, which satisfies $\phi_x(x,y) = y = M(x,y)$ and $\phi_y(x,y) = x = M(x,y) \ \forall (x,y) \in D$. Then ϕ is a potential for $ydx + xdy$ in this case, by definition. Hence, $ydx + xdy = 0$ is exact by definition. Hence, I here is an integrating factor for this ode, by definition as well.

Exercise 40. In the above example, suppose instead that $I : D \to \mathbb{R}$ is given by $I(x,y) = y$. Show that I is not an integrating factor for $ydx + xdy = 0$.

Theorem 12.11 (Integrating Factor Solution Invariance Theorem)**.** *Suppose that J is an open interval in $\mathbb{R}$ and that $y : J \to \mathbb{R}$ is differentiable and unknown. Let*

$$\{(x, y(x)) : x \in J\} \subset D \subset \mathbb{R}^2.$$

Suppose that $M, N : D \to \mathbb{R}$. Suppose that I is an integrating factor for $M(x,y)dx + N(x,y)dy = 0$, and that an equation or system of equations is a solution to $I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$. Then this solution is also a solution of $M(x,y)dx + N(x,y)dy = 0$.

Example 12.12. We have showed that $I : \{(x,y) \in \mathbb{R}^2 : x \neq 0\} \to \mathbb{R}$ given by $I(x,y) = x$ is an integrating factor for

$$\frac{y}{x}dx + dy = 0. \tag{6}$$

That is, we have that

$$(x)\frac{y}{x} + xdx = 0$$

is exact. That is,

$$ydx + xdy = 0$$

is exact. We also showed that any solution of this equation can be written in the form

$$xy = c$$

for some $c \in \mathbb{R}$. The Integrating Factor Solution Invariance Theorem guarantees that this general solution is also a general solution of Equation (6) above.

Exercise 41. Suppose that $D = \{(x,y) \in \mathbb{R}^2 : x, y \neq 0\}$, and let $M, N : D \to \mathbb{R}$ be given by

$$M(x,y) = xy^2 + 4x^2y \text{ and } N(x,y) = 3x^2y + 4x^3.$$

(a) Prove that $M(x,y)dx + N(x,y)dy = 0$ is not exact.

(b) Prove that $I : D \to \mathbb{R}$ given by $I(x,y) = y/x$ is an integrating factor for the above ode.

(c) Solve $I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$.

(d) Solve $M(x,y)dx + N(x,y)dy = 0$ using Part (c), among other things.

Research Project: Prove the Integrating Factor Solution Invariance Theorem.

Theorem 12.13 (Integrating Factor PDE Characterization Theorem)**.** *Suppose that J is an open interval in $\mathbb{R}$ and that $y : J \to \mathbb{R}$ is differentiable and unknown. Let*

$$\{(x, y(x)) : x \in J\} \subset D \subset \mathbb{R}^2.$$

Suppose that $M, N : D \to \mathbb{R}$. Then I is an integrating factor of $M(x,y)dx + N(x,y)dy = 0$ iff I is a solution to

$$N(x,y)I_x(x,y) - M(x,y)I_y = [(M_y(x,y) - N_x(x,y)]I(x,y). \tag{7}$$

Example 12.14. In the above example, recall that we proved that $I : D \to \mathbb{R}$ given by $I(x,y) = x$ is an integrating factor for

$$\frac{y}{x}dx - 1dy = 0,$$

assuming that $y : \mathbb{R} \to \mathbb{R}$ is unknown and differentiable. Recalling here that M and N are respectively given by $M(x,y) = y$ and $N(x,y) = x$, we have, by the Integrating Factor Characterization Theorem that Equation (7) holds. In other words, we have that I must be a solution to

$$1I_x(x,y) - \frac{y}{x}I_y = \left(\frac{1}{x} - 0\right) I.$$

This pde can be more simply written as

$$I_x - \frac{y}{x}I_y = \frac{I}{x}.$$

The problem with this theorem is that this course is not designed to handle pde's. Such equations are considered in an upper division course on pde's. Thus we will not have much use for the Integrating Factor PDE Characterization Theorem.

Research Project: Prove the theorem.

The following theorem is much more useful for us:

Theorem 12.15 (One Variable Integration Factor Generation Theorem)**.** *Suppose that J is an open interval in $\mathbb{R}$ and that $y : J \to \mathbb{R}$ is differentiable and unknown. Let*

$$\{(x, y(x)) : x \in J\} \subset D \subset \mathbb{R}^2.$$

Suppose that $M, N : D \to \mathbb{R}$. Then the following two statements hold:

(1) $\exists f : J \to \mathbb{R}$ *such that* $\forall (x,y) \in D$, *we have that*

$$f(x) = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)},$$

iff $I := e^F$ *is an integrating factor of* $M(x,y)dx + N(x,y)dy = 0$ *for some* $F \in \int f(x)dx$.

(2) $\exists g : Rng(y) \to \mathbb{R}$ *such that* $\forall (x,y) \in D$, *we have that*

$$g(y) = \frac{M_y(x,y) - N_x(x,y)}{M(x,y)},$$

iff $I := e^{-G}$ *is an integrating factor of* $M(x,y)dx + N(x,y)dy = 0$ *for some* $G \in \int g(x)dx$.

Example 12.16. Suppose that $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$, and suppose that $M, N : D \to \mathbb{R}$ are given by

$$M(x,y) = 5y \text{ and } N(x,y) = x^2.$$

Let's use the One-variable Integration Factor Generation Theorem to find an integrating factor for $M(x,y)dx + N(x,y)dy = 0$. That is, let's find an integrating factor, using that Theorem, for

$$5ydx + x^2dy = 0. \tag{8}$$

Now we have that $\forall (x,y) \in \mathbb{R}^2$,

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{\frac{\partial}{\partial y}(5y) - \frac{\partial}{\partial x}(x^2)}{x^2} = \frac{5 - 2x}{x^2},$$

which is a real number if $(x,y) \in D$. Define $f : \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$ by

$$f(x) = \frac{5 - 2x}{x^2}.$$

Then we have that

$$f(x) = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)}.$$

Hence, the One Variable Integrating Factor Generation Theorem guarantees that if $F \in \int f(x)dx$, then e^F is an integrating factor for Equation (8). Now we have that

$$\begin{aligned} \int f(x)dx &= \int \frac{5-2x}{x^2}dx \\ &= \int \left(\frac{5}{x^2} - \frac{2x}{x^2}\right)dx \\ &= \int \left(5x^{-2} - 2x^{-1}\right)dx \\ &= -5x^{-1} - 2\ln x + C. \end{aligned}$$

Thus $F : \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$ given by

$$F(x) = \frac{-5}{x} - 2\ln x$$

is an element of $\int f(x)dx$. Furthermore, e^F is an integrating factor for Equation (8).

Exercise 42. Solve Equation (8).

Exercise 43. Use the One-variable Integrating Factor Generation Theorem to find an integrating factor for

$$(2x - y^2)dx + 3xydy = 0.$$

Then solve the equation using the Integrating Factor Solution Invariance Theorem.

Exercise 44. Prove the theorem.

Example 12.17. Let's use the One-variable Integration Factor Generation Theorem to find an integrating factor for $M(x,y)dx + N(x,y)dy = 0$ if D is the first quadrant in $\mathbb{R}^2$ and $M, N : D \to \mathbb{R}$ is given by

$$M(x,y) = 3xy \text{ and } N(x,y) = x^2.$$

If $(x,y) \in D$, then we have that

$$\frac{M_y - N_x}{M(x,y)} = \frac{\frac{\partial}{\partial y}(3xy) - \frac{\partial}{\partial x}(x^2)}{3xy} = \frac{3x - 2x}{3xy} = \frac{x}{3xy} = \frac{1}{3y}.$$

Now let g :Rng$(y) \to \mathbb{R}$ be given by

$$g(y) = \frac{1}{3y}.$$

Then we have that

$$\int g(y)dy = \int \frac{1}{3y}dy = \frac{1}{3}\int \frac{1}{y}dy = \frac{1}{3}\ln|y| + C.$$

Thus G :Rng$(y) \to \mathbb{R}$ given by

$$G(y) = \frac{1}{3}\ln|y|$$

is in $\int g(y)dy$. The One-variable Integration Factor Generation Theorem guarantees that e^{-G} is an integrating factor for $3xydx + x^2dy = 0$. Thus we can let $I : D \to \mathbb{R}$ be given by

$$I(x,y) = e^{-\frac{1}{3}\ln|y|} = e^{\ln|y|^{-1/3}} = |y|^{-1/3} = \frac{1}{\sqrt[3]{y}},$$

and we can now conclude that I is an integrating factor for $M(x,y)dx + N(x,y)dy = 0$ here.

Exercise 45. Solve $M(x,y)dx + N(x,y)dy = 0$ in the above example, now that you have an integrating factor for it.

Exercise 46. Assume that (x,y) is always in Quadrant I. Solve

$$xydx + (5y - x^2)dy = 0.$$

13. Numerical Methods Including Euler's Method for Approximately Solving First-Order IVP's

Definition 13.1 (Numerical approximation of a function)**.** Suppose that $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$. We call a list of ordered pairs $(x_0, y[x_0]), (x_1, y[x_1]), \ldots, (x_n, y[x_n])$ a *numerical approximation* to f iff $y(x_j)$ is an approximate value of $f(x_j)$ $\forall j = 0, 1, 2, \ldots, n$.

Example 13.2. Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = x^2$, and let $x_0 = 0$, $x_1 = .1$, and $x_2 = .2$. We already are given that $y(x_0) = y(0) = 1$. Let $\Delta x = .1$, and recall that

$$\begin{aligned} f(x_0 + \Delta x) &\sim f(x_0) + f'(x_0)\Delta x \\ &= 0^2 + [2x_0]_{x_0=0}(.1) \\ &= 0 \end{aligned}$$

gives an approximate value of $f(0.1)$. In the same way,

$$\begin{aligned} f(x_1 + \Delta x) &\sim f(x_1) + f'(x_1)\Delta x \\ &= .1^2 + [2(x)]_{x=.1}(.1) \\ &= .01 + 2(.01) = .03 \end{aligned}$$

gives an approximate value of $f(0.2)$. In the same way,

$$\begin{aligned} f(x_2 + \Delta x) &\sim f(x_2) + f'(x_2)\Delta x \\ &= .03^2 + [2(x)]_{x=.03}(.1) \\ &= .0009 + 2(.003) = .0069 \end{aligned}$$

gives an approximate value of $f(0.3)$. Therefore the collection of ordered pairs

$$\{(0, 0), (0.1, 0), (0.2, .03), (0.3, .0069)\}$$

can be thought of as a numerical approximation to f.

Exercise 47. Find a numerical approximation to the square root function using 5 points in $[0, 10]$ and ensuring that the approximation's y-values are accurate to within one decimal place of the actual y-value.

Definition 13.3 (Numerical solution to an ode)**.** We call a list of points $x_0, x_1, x_2, \ldots, x_n$ together with a list of values $y(x_0), y(x_1), \ldots, y(x_n)$ that is a numerical approximation to f a *numerical solution* to an ode in an unknown function y iff f is a solution to the ode.

Example 13.4. Consider the ode $y' = 2x$ in the unknown function y. Notice that f in the example above is a solution to this ode. Therefore, the set of ordered pairs

$$\{(0, 0), (0.1, 0), (0.2, 0.3), (0.3, .0069)\}$$

can be thought of as a numerical solution to the DE, since this list is a numerical approximation of f.

Exercise 48. Find a numerical solution to the de $y' = x^{-1/2}$ using x-values

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

and a first-order Taylor polynomial of order 1 for the solution function.

Definition 13.5 (Euler's method for first-order IVP's)**.** Suppose that $g : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ contains the inside portion of a circle centered at (x_0, y_0). Assume that $\Delta x > 0$ and that $x_j := x_0 + j\Delta x \in \text{Dom}(y)\ \forall j = 0, 1, 2, \ldots, n$. Then the determination of the list of points $x_0, x_1, x_2, \ldots x_n$ together with outputs $L(x_0), L(x_1), L(x_2), \ldots L(x_n)$, where L is given by $L(x_0) = y_0$ and for $j = 1, 2, \ldots, n$,

$$L(x_j) = y_{j-1} + g(x_{j-1}, y_{j-1})\Delta x$$

is called *Euler's method* for obtaining the numerical solution

$$(x_0, L[x_0]), (x_1, L[x_1]), \ldots, (x_n, L[x_n])$$

to the IVP $y' = g(x, y)$, $y(x_0) = y_0$.

Example 13.6. Consider the IVP

$$y' = 1 - 2y, \quad y(0) = 1.$$

Let's use Euler's method to find a numerical solution to this IVP by using $x_0 = 0$, $\Delta x = .1$, and $n = 3$. We obtain

$$\begin{aligned} y_1 &= y_0 + g(x_0, y_0)\Delta x \\ &= 1 + (1 - 2y_0)(.1) \\ &= 1 + (1 - 2)(.1) \\ &= 0.9, \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + g(x_1, y_1)\Delta x \\ &= 0.9 + (1 - 2[.9])(0.1) \\ &= 0.82, \text{ and} \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + g(x_2, y_2)\Delta x \\ &= 0.82 + (1 - 2[0.82])(.1) \\ &= .756. \end{aligned}$$

Therefore, $(0, 1), (0.1, 0.9), (0.2, 0.82), (0.3, 0.756)$ is a numerical solution to the IVP using Euler's method.

Let's check to see how close our values are to the actual function values of the exact solution. The reader should keep in mind that some DE's do not have known, exact solutions. We have simply used an IVP that we do have the exact solution for, for illustrative purposes.

Note that $y = .5 + .5e^{-2x}$ is a solution to the IVP, since in this case, $y(0) = .5 + .5e^{-2(0)} = .5 + .5 = 1$,

$$y' = .5e^{-2x}(-2) = -e^{-2x}, \text{ and}$$
$$1 - 2y = 1 - 2(.5 + .5e^{-2x}) = -e^{-2x} = y'.$$

For this solution, the reader can see that

$$y(0) = .5 + .5e^{-2(0)} = 1, y(.1) = .5 + .5e^{-2(.1)} \sim .90936577,$$
$$y(.2) = .5 + .5e^{-2(.2)} \sim .83560023, \text{ and}$$
$$y(.3) = .5 + .5e^{-2(.3)} \sim .774405818.$$

The reader should compare these "exact" (not really exact due to calculator rounding, of course) to the Euler-approximated y-values 1, 0.9, 0.82, and 0.756. (Not a bad approximation, huh?)

Exercise 49. Use Euler's method with $n = 5$ and $\Delta x = 0.2$ to obtain an approximate solution to the IVP

$$y' = x + y, \ \ y(-1) = 0.$$

Then, solve the IVP, calculate $y(x_j)$ for each $j \in \{0, 1, 2, \ldots, n\}$, and approximate the error made by using Euler's method, for each one of these j's.

Definition 13.7 (Heun's Improved Euler's method for first-order IVP's)**.** Suppose that $g : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ contains the inside portion of a circle centered at (x_0, y_0). Assume that $\Delta x > 0$ and that $x_j := x_0 + j\Delta x \in \mathrm{Dom}(y)\ \forall j = 0, 1, 2, \ldots, n$. Then the determination of the list of points $x_0, x_1, x_2, \ldots x_n$ together with outputs $L(x_0), L(x_1), L(x_2), \ldots L(x_n)$, where L is given by $L(x_0) = y_0$ and for $j = 1, 2, \ldots, n$,

$$\begin{aligned} z_j &= y_{j-1} + g(x_{j-1}, y_{j-1})\Delta x, \text{ and} \\ L(x_j) &= y_{j-1} + \left[\frac{g(x_{j-1}, y_{j-1}) + g(x_j, z_j)}{2}\right]\Delta x \end{aligned}$$

is called the *Heun's improved Euler's method* for obtaining the numerical solution

$$(x_0, L[x_0]), (x_1, L[x_1]), \ldots, (x_n, L[x_n])$$

to the IVP $y' = g(x, y)$, $y(x_0) = y_0$.

Example 13.8. Let's redo the Example 13.6 using Heun's Improved Euler's Method. First, recalling there that $x_0 = 0$, $y_0 = 1$, and $x_1 = \Delta x = 0.1$, we obtain that

$$z_1 = y_0 + g(x_0, y_0)\Delta x = 1 + (1 - 2y_0)(.1) = 1 + (1 - 2[1]).1 = .9.$$

We also obtain that

$$\begin{aligned} L(x_1) &= y_0 + \left[\frac{g(x_0, y_0) + g(x_1, z_1)}{2}\right](.1) \\ &= 1 + \left[\frac{-1 + (1 - 2z_1)}{2}\right](.1) \\ &= 1 + \left[\frac{-1 + (1 - 2[.9])}{2}\right](.1) \\ &= 0.91. \end{aligned}$$

Therefore, we take y_1 to be 0.91. We then obtain that

$$z_2 = y_1 + g(x_1, y_1)(0.1) = 0.91 + (1 - 2y_1)(.1) = .91 + (1 - 2[0.91])(.1) = 0.828,$$

thus giving y_2 approximated by

$$\begin{aligned} L(x_2) &= y_1 + \left[\frac{g(x_1, y_1) + g(x_2, z_2)}{2}\right](.1) \\ &= 0.91 + \left[\frac{-0.82 + 1 - 2[0.828]}{2}\right](.1) \\ &= 0.8362. \end{aligned}$$

We have that

$$z_3 = y_2 + g(x_2, y_2)\Delta x = 0.8362 + (1 - 2[0.8362])(.1) = 0.76896$$

Therefore, y_3 is approximated by

$$\begin{aligned} L(x_3) &= y_2 + \left[\frac{g(x_2, y_2) + g(x_3, z_3)}{2}\right](.1) \\ &= .8362 + \left[\frac{1 - 2(.8362) + 1 - 2(.76966)}{2}\right](.1) \\ &= 0.775684, \end{aligned}$$

and our numerical solution using the Improved Euler's Method is

$$(0, 1), (.1, .86), (.2, .7952), (.3, .7630445568).$$

Remark: Often, Euler's Improved Method is also called "Euler's modified method" or "Heun's Method."

Exercise 50. Find the approximate error for each of the numerically obtained y-values in the above example.

Exercise 51. Redo Exercise 49 using Heun's Improved Euler's method.

Definition 13.9 (Order 4 Runge-Kutta Method)**.** Suppose that $g : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ contains the inside portion of a circle centered at (x_0, y_0). Assume that $\Delta x > 0$ and that $x_j := x_0 + j\Delta x \in \text{Dom}(y)\ \forall j = 0, 1, 2, \ldots, n$. Then the determination of the list of points $x_0, x_1, x_2, \ldots x_n$ together with outputs $L(x_0), L(x_1), L(x_2), \ldots L(x_n)$, where L is given by $L(x_0) = y_0$, for $j = 1, 2, \ldots, n$, and for the rest of the j's, $L(x_j)$ is given by y_j, where

$$y_{j+1} = y_j + \frac{1}{6}(k_{1,j} + 2k_{2,j} + 2k_{3,j} + k_{4,j}),$$

and the following equations hold:

$$k_{1,j} = g(x_0, y_0)\Delta x,\ \ k_{2,j} = g(x_j{+}0.5\Delta x, y_j{+}0.5k_{1,j})\Delta x,\ \ k_{3,j} = g(x_j{+}0.5\Delta x, y_j{+}0.5k_{2,j})\Delta x \quad \text{and}$$

$$k_{4,j} = g(x_{j+1}, y_j + k_{3,j})\Delta x.$$

is called the *Order* 4 *Runge-Kutta method* for numerically approximating a solution to the IVP

$$y' = g(x, y), y(x_0) = y_0$$

with unknown function y.

Example 13.10. Let's use the Order 4 Runge-Kutta method with $n = 3$ and $\Delta x = 1$ to find a numerical approximation to the solution the following IVP in the unknown y:

$$y' = 3, \ \ y(0) = 0.$$

Here, we have that $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. Also, $g(x, y)$ here is 3. Note that here, we have

$$k_{1,1} = k_{1,2} = k_{1,3} = k_{1,4} = g(x_0, y_0)\Delta x = g(0, 0)(1) = 3(3) = 3.$$

Moreover, we have that since g is the constant function 3 and $\Delta x = 1$, we must have that for all $i \in \{2, 3, 4\}$ and $j \in \{2, 3, 4\}$, $k_{i,j} = 3$. Since $y_0 = y(0) = 0$, we have that $y_1 = 0+1/6(3+2[3]+2[3]+3) = 3$. We also obtain $y_2 = 3+1/6(18) = 6$, $y_3 = 6+1/6(18) = 9$. Thus our numerical solution here is

$$(0, 0), (1, 3), (2, 6), (3, 9)).$$

Exercise 52. Solve the IVP $y' = 3$, $y(0) = 0$ and show that the error in the numerical approximation obtained above is 0.

Exercise 53. Redo Exercise 49 using, instead, the Order 4 Runge-Kutta method, using $\Delta x = 0.1$ with $n = 10$. How accurate is your numerical solution compared with the solutions obtained by

(a) Euler's Method?

(b) Heun's Improved Euler's Method?

14. Solving some second-order ODE's

In this section, we present a reduction theorem that allows us to reduce a second-order ode in which the second derivative is isolated, into a system of two ode's, each of which is only first order.

Theorem 14.1 (Second-Order to First-Order System Reduction Theorem)**.** *Suppose that J is an interval in $\mathbb{R}$, and assume that $y : J \to \mathbb{R}$ is differentiable and unknown.*

Suppose that

$$\begin{aligned} \{(x, y(x), y'(x)) : x \in J\} \ &\subset \ D \subset \mathbb{R}^3, \\ \{(x, y'(x)) : x \in J\} \ &\subset \ E \subset \mathbb{R}^2, \ \textit{and} \\ \{(y(x), y'(x)) : x \in J\} \ &\subset \ B \subset \mathbb{R}^2. \end{aligned}$$

Let $F : D \to \mathbb{R}$, $G : E \to \mathbb{R}$, and $H : B \to \mathbb{R}$. Then the following statements hold:

(A) The system $dy/dx = v$, $dv/dx = F(x, y, v)$ is a solution to $y'' = F(x, y, y')$, and

(B) The system $y' = v$, $v' = G(x, v)$ is a solution to $y'' = G(x, y')$,

(C) The system $dy/dx = v$, $dv/dy = H(y, v)$ is a solution to $y'' = H(y, y')$.

Example 14.2. As an easy first example, suppose that $y : \mathbb{R} \to \mathbb{R}$ is unknown. Consider $y'' = 5$. Here, let $F : \mathbb{R}^3 \to \mathbb{R}$ be given by $F(x, y, z) = 5$. By Part (A) of the Reduction Theorem above, we have that $dy/dx = v$, $dv/dx = F(x, y, v)$ is a solution of $y'' = 5$. Let's focus on $dv/dx = F(x, y, v)$, which can be rewritten as $dv/dx = 5$. A general solution to this simple ode is $v = 5x + C$. Thus $dy/dx = 5x + C$ is a solution to $y'' = 5$. However, we'd like to find y, if possible, and that is very easy here: Indeed, we have that

$$y = \frac{5x^2}{2} + Cx + D,$$

is a general solution to $y'' = 5$, where C and D are any real constants.

Exercise 54. Use the Reduction Theorem to obtain a system of equations that together form a solution to $y'' = -4y$. Then find an explicit formula for the solution y.

Example 14.3. Let $D = \mathbb{R}^2$, and define $F : \mathbb{R}^2 \to \mathbb{R}$ be given by $F(x, y) = x + y$. Let's solve $y'' = F(x, y')$ using the above Reduction Theorem. That is, let's solve $y'' = x + y'$. By Part (B) of the theorem, the system $dy/dx = v$, $dv/dx = x + y'$ is a solution of $y'' = x + y'$. Let's see if we can do better by using this system of equations to generate an explicit solution y, where y is currently unknown and has domain $\mathbb{R}$ and is real-valued. We focus first on solving $dv/dx = x + y'$, which is equivalent to $dv/dx = x + v$, or, better yet, $dv/dx - 1v = x$, which is standard form for a first order linear ode. Since this ode is not separable, as we showed in an earlier section with v replaced by y, we'll use the Integrating Factor Theorem for Monic, First Order Linear ODE's. Here, $p(x) = -1$, and an antiderivative P for p is given by $P(x) = -x$. We let $w(x) = e^{-x}$ and $q(x) = x$, obtaining that

$$\int w(x)q(x)dx = \int e^{-x}(x)dx = \int xe^{-x}dx.$$

Letting $u = x$ and $dr = e^{-x}dx$, we obtain that $du = dx$ and that $r = -e^{-x} + C$. Thus we have that the above integral is $ur - \int rdu = -xe^{-x} - \int(-e^{-x})dx = -xe^{-x} + \int e^{-x}dx = xe^{-x} - e^{-x} + C$. Thus $G : \mathbb{R} \to \mathbb{R}$ given by $G(x) = (x - 1)e^{-x}$, and we have that

$$v = \frac{1}{e^{-x}}[(x - 1)e^{-x} + C],$$

and this is a general solution to $dv/dx - 1v = x$. Let's simplify the right side of the above equation. We then obtain that

$$v = x - 1 + Ce^x$$

is the general solution of $dv/dx = x + v$. Thus, we have that $dy/dx = x - 1 + Ce^x$, which implies that

$$y = \frac{x^2}{2} - x - Ce^x$$

is a general solution of $y'' = x + y'$. Since C is arbitary, we can change the $-$ sign in front of it above to $+$ without loss of generality, thus obtaining that

$$y = \frac{x^2}{2} - x + Ce^x.$$

Exercise 55. Check, by simple substitution, that the above general solution really solves $y'' = x + y'$.

Exercise 56. Suppose that $y : \mathbb{R} \to \mathbb{R}$ is unknown, and solve $y'' = x - y'$.

Example 14.4. In this example, we'll apply Part (C) of the Reduction Theorem. Let $y : \mathbb{R} \to \mathbb{R}$ be unknown and differentiable. Let's solve $y'' = y + y'$. Here we let $H : \mathbb{R}^2 \to \mathbb{R}$ be given by $H(x, y) = x + y$, and we solve $y'' = H(y, y')$. By Part (C) of the Reduction Theorem above, we have that $dy/dx = v$, $dv/dy = y + v$ is a solution to this ode. In the previous example, we already showed that $dv/dx = x + v$ has general solution $v = x - 1 + Ce^x$. Thus in this example, we have that $v = y - 1 + Ce^y$. That is, we have that $y' = y - 1 + Ce^y$. This ode is separable, but involves finding the indefinite integral of the expression $1/(x - 1 + Ce^x)$, which methods that the reader has seen so far do not adequately handle. Thus, are only option is to find an approximate solution for various choices of C. The reader is strongly urged to complete the following exercise, which continues this line of thought:

Exercise 57. Find a numerical solution to $y' = y - 1 + Ce^y$ in the special case that $C = 1$; that is, find approximate y-values corresponding to $x_k = 0.1k\Delta x$ for $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ using
(a) Euler's method;
(b) Heun's method;
(c) the Order 4 Runge-Kutta method.

In all three cases, write your numerical solution as a collection of 11 ordered pairs, thus providing a numerical solution to the second order ode in the above example.

Exercise 58. Prove the above theorem.

15. Basic Matrix Notation

The methods in the previous section only work for *some* second-order ode's. Other types of ode's will only be solvable if we use matrix theory and more general theory called linear algebra, which involves the notion of a vector space, to be explained later. In the next three chapters, we'll need to take a break from ode's and develop tools from linear algebra. This chapter deals with complex systems of equations and (augmented) matrices, and the next chapter deals with determinants. After that chapter, we'll consider vector spaces, at which point we'll be ready to study eigenvalues and eigenvectors, which are needed for solving higher-order systems of linear ode's. Most of this chapter should be review, the only new thing being that everything you learned about matrices in precalculus is also true when the matrix entries are complex numbers rather than merely real.

Definition 15.1. If X is a non-empty set, then
1. $M_{m\times n}(X)$ denotes {all $m \times n$ matrices whose entries are in X}.
2. $M_n(X)$ denotes $M_{n\times n}(X)$, which is referred to as the collection of *square matrices of order n*.

Example 15.2.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \in M_{3\times 2}(\mathbb{R}).$$

Denoting this matrix by A, we have that $a_{1,1} = 1$, $a_{1,2} = 2$, $a_{2,1} = 3$, $a_{2,2} = 4$, $a_{3,1} = 5$, and $a_{3,2} = 6$.

Exercise 59. Give an example of a matrix in $A \in M_{5\times 7}(\mathbb{C})$ and an example of a matrix in $B \in M_3(\mathbb{R})$. What is $a_{4,3}$ in your matrix A? What is $b_{2,1}$ in your matrix B?

Definition 15.3. If X is a set, and A has *index form* $(a_{i,j})_{i=1,j=1}^{m,n} \in M_{m\times n}(X)$, then we define the *transpose* of A by

$$A^T = (b_{i,j})_{i=1,j=1}^{n,m},$$

where $\forall i \in \{1, 2, 3 \dots, n\}$ and $j \in \{1, 2, 3, \dots, m\}$, $b_{i,j} := a_{j,i}$.

Example 15.4. Let's find A^T if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

The indexed entries of A are $a_{1,1} = 1$, $a_{1,2} = 2$, $a_{2,1} = 3$, $a_{2,2} = 4$, $a_{3,1} = 5$, and $a_{3,2} = 6$. By definition, we have that

$$A^T = (b_{i,j})_{i=1,j=1}^{n,m}$$

where $\forall i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$, we have $b_{i,j} = a_{j,i}$. Thus

$$\begin{aligned} A^T &= \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{bmatrix} \\ &= \begin{bmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}. \end{aligned}$$

Exercise 60. Find $(I_4)^T$ if I_4 is the 4×4 identity matrix. Prove your answer.

Exercise 61. Find the transpose of the matrix obtained in the previous example; that is, find $(A^T)^T$.

Definition 15.5. If X is any non-empty set, and $m, n \in \mathbb{N}$, then we call $A \in M_{m\times n}(X)$ *symmetric* iff $m = n$ and $A^T = A$.

Example 15.6. A in the last example is not symmetric, since

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = A$$

.

Example 15.7. Recall that the 2×2 identity matrix is given by

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let's prove that I_2 is symmetric. Here, if we let the index form of I_2 be given by $(a_{i,j})_{i,j=1}^{2}$, then we have

$$a_{1,1} = 1, a_{1,2} = 0$$

$$a_{2,1} = 0, \text{ and } a_{2,2} = 1.$$

Therefore, $I_2^T = (b_{i,j})_{i,j=1}^2$, where

$$b_{1,1} = a_{1,1} = 1, b_{1,2} = a_{2,1} = 0$$
$$b_{2,1} = a_{1,2} = 0, b_{2,2} = a_{2,2} = 1$$

Hence, $I_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. Therefore, I_2 is symmetric, by definition.

Exercise 62. Give an example of a square, real matrix of order 3 that has at least three different entries and is symmetric. Prove that your matrix is symmetric.

Exercise 63. Show that if $A \in M_{m\times n}(X)$, where X is a non-empty set and $m \neq n$, then A is not symmetric.

Definition 15.8. If $X = \mathbb{R}$, then we denote $M_{m\times n}(X)$ by $M_{m\times n}$ and $M_n(X)$ by M_n

Example 15.9. $M_{3\times 2} = \{\text{all } 3 \times 2 \text{ real matrices}\}$

Exercise 64. Give an example of a matrix that is in $M_{4\times 9}$.

Example 15.10. M_3=$\{\text{all } 3 \times 3 \text{ real matrices}\}$

Exercise 65. Give an example of a matrix that is in M_6.

Definition 15.11. Let $m, n \in \mathbb{N}$. We say that $A \in M_{m\times n}$ is *skew-symmetric* iff $m = n$ and $A^T = -A$

Example 15.12. Consider

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let's prove that this matrix is skew-symmetric. Writing $A = (a_{i,j})_{i,j=1}^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we observe that

$$a_{1,1} = 1, a_{1,2} = 1$$
$$a_{2,1} = -1, \text{ and } a_{2,2} = 0.$$

Then by definition of transpose,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = (b_{i,j})_{i,j=1}^2$$

where $\forall i, j \in \{1, 2\}$, we have that $b_{i,j} = a_{j,i}$. Now

$$b_{1,1} = a_{1,1} = 0, b_{1,2} = a_{2,1} = -1$$
$$b_{2,1} = a_{1,2} = 1, \text{ and } b_{2,2} = a_{2,2} = 0.$$

So

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1(0) & -1(1) \\ -1(-1) & -1(0) \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew-symmetric, by definition.

Exercise 66. Find an example of a skew symmetric matrix $A \in M_1$. Prove that your choice of A is skew-symmetric. *Remark:* It is highly advised that you prove every answer that you obtain to any exercises in these notes, so we will no longer mention after the two exercises below this one, that you should prove your answer. When would you NOT want to know you are correct? The answer is "NEVER!"

Exercise 67. Find an example of a skew-symmetric matrix $A \in M_3$. Prove that your choice of A is skew-symmetric.

Exercise 68. Find an example of a skew-symmetric matrix $A \in M_4(\mathbb{C})$ such that A contains has no entries that are 0 and such that A has at least 6 different entries, at least five of which are complex and non-real. Prove that your obtained matrix A is skew-symmetric.

Definition 15.13. We call $A = (a_{i,j})_{i=1,j=1}^{m,n} \in M_{m\times n}(\mathbb{C})$

(1) *lower triangular* iff $a_{i,j} = 0$ whenever $i < j$,

(2) *upper triangular* iff $a_{i,j} = 0$ whenever $i > j$,

(3) *diagonal* iff $a_{i,j} = 0 \ \forall i \neq j$,

(4) *unit lower triangular* iff A is lower triangular and $a_{i,i} = 1 \ \forall i \in \{1, 2, 3, \ldots, m\}$, and

(5) *unit upper triangular* iff A is upper triangular and $a_{j,j} = 1 \ \forall j \in \{1, 2, 3, \ldots, n\}$.

Example 15.14. Let's prove that I_2 is lower triangular, upper triangular, diagonal, unit lower triangular, and unit upper triangular. Then we'll prove that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is none of these.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has double-index form $(a_{i,j})_{i,j=1}^{2}$, where

$$a_{1,1} = 1, a_{1,2} = 0$$

$$a_{2,1} = 0, \text{ and } a_{2,2} = 1.$$

Let $i, j \in \{1, 2\}$ and assume that $i > j$. Then $i = 2$ and $j = 1$, from which it follows that $a_{i,j} = a_{2,1} = 0$. Thus we've shown that if $i > j$, $a_{i,j} = 0$. Hence, I_2 is upper triangular by definition.

Next, assume that $i, j \in \{1, 2\}$ and $i < j$. Then we must have that $i = 1$ and $j = 2$. Hence, $a_{i,j} = a_{1,2} = 0$. It follows that I_2 is lower triangular by definition. Since $a_{1,1} = a_{2,2} = 1$, I_2 is also unit lower triangular and unit upper triangular by definition.

Assume that $i, j \in \{1, 2\}$ and that $i \neq j$. Then, either $i < j$ or $i > j$. We already showed that $a_{i,j} = 0$ in both cases, so $a_{i,j} = 0 \ \forall i \neq j$. Hence, I_2 is diagonal by definition.

Now we prove our claim about

$$B = (b_{i,j})_{i,j=1}^{2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

First, we note that $b_{2,1} = 3 \neq 0$. Thus it is not the case that $b_{i,j} = 0$ whenever $i > j$. Hence, by definition, B is not upper triangular.

Now $b_{2,1} = 2 \neq 0$, so it's not the case that $b_{i,j} = 0$ whenever $i < j$. Thus by definition, B is not lower triangular.

Also, since $b_{1,2} \neq 0$, then it is not the case that $b_{i,j} = 0 \ \forall i \neq j$ Hence, B is not diagonal either. Furthermore, B is neither unit lower triangular nor unit upper triangular because it is neither lower triangular nor upper triangular.

Exercise 69. Find an example of a real matrix that is upper triangular but not unit upper triangular.

Exercise 70. Find an example of a real matrix that is lower triangular but not unit lower triangular.

Exercise 71. Find a 2×2 real matrix that is diagonal but not I_2.

Exercise 72. Find an example of a matrix with at least 3 rows and at least 3 columns that is upper triangular but not unit upper triangular.

Exercise 73. Find an example of a matrix with at least 3 rows and at least 3 columns that is lower triangular but not unit lower triangular.

Definition 15.15. If $A = (a_{i,j})_{i=1,j=1}^{m,n} \in M_{m\times n}(x)$, where $m, n \in \mathbb{N}$ and X is a non-empty set, then $(a_{1,1}, a_{2,2}, \ldots, a_{m,m})$ is called the *main diagonal* of A.

Example 15.16. In I_2, the main diagonal, by definition, is $(a_{1,1}, a_{2,2}) = (1, 1)$, since $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 74. Find the main diagonal of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Remember that you should prove all answers to all exercises in these notes.

Definition 15.17. We denote $M_{1\times n}(\mathbb{R})$ by $\mathbb{R}^n$ and $M_{1\times n}(\mathbb{C})$ by $\mathbb{C}^n$. More generally, we denote $M_{1\times n}(X)$ by X^n, $\forall$ non-empty sets X. Furthermore, we sometimes denote $[a_1, a_2, \ldots, a_n]$ by $(a_1, a_2, \ldots, a_n)$.

Example 15.18. $M_{1\times 3}(\mathbb{R}) = \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$. Also, we have that $(-1, 3, 5) = [-1 \ \ 3 \ \ 5] \in M_{1\times 3}(\mathbb{R})$. Since $\mathbb{R} \subset \mathbb{C}$, we have that $(1, -3, 5) \in \mathbb{C}^3$ as well.

Exercise 75. Give an example of an element of $\mathbb{R}^6$. Use the above definition to confirm that your example is indeed in $\mathbb{R}^6$.

Exercise 76. Give an example of an element of $\mathbb{C}^5$ that is not in $\mathbb{R}^5$, and use the above definition to confirm your answer.

Definition 15.19. We denote by $M_{n\times 1}(\mathbb{R})$ by $\mathbb{V}_n$, and we denote $M_{n\times 1}(\mathbb{C})$ by $\mathbb{V}_n^{\mathbb{C}}$. $\mathbb{V}_n$ is called the *vectors in* $\mathbb{R}^n$. Also, $\mathbb{V}_n^{\mathbb{C}}$ is called the *vectors in* $\mathbb{C}^n$.

Example 15.20. $\mathbb{V}_2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$, and $\mathbb{V}_2^{\mathbb{C}} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{C} \right\}$.

Example 15.21. $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \in \mathbb{V}_3, \mathbb{V}_3^{\mathbb{C}}$.

Exercise 77. Give an example of a vector in $\mathbb{V}_5$.

Exercise 78. Give an example of a vector in $\mathbb{V}_7^{\mathbb{C}}$ that is not in $\mathbb{V}_7$.

Exercise 79. Show that $\mathbb{V}_n \neq \mathbb{R}^n$ unless $n = 1$. Prove that this statement is also true if $\mathbb{V}_n$ is replaced by $\mathbb{V}_n^{\mathbb{C}}$.

Definition 15.22. We will denote the collection of all matrices by $\mathcal{M}$.

Example 15.23. $\begin{bmatrix} L & \Rightarrow \\ 5 & :) \end{bmatrix} \in \mathcal{M}$.

Exercise 80. Find an example of an element of $\mathcal{M}$ whose entries are only names of you, your children if you have any, or any of your ancestors, which you must have at least two of, unless you are a weird space alien!

Definition 15.24 (Matrix function)**.** f is called a *matrix function* iff the codomain of f is a subset of $\mathcal{M}$, but such that no element of the codomain is a matrix with only one entry in it.

Example 15.25. Suppose that $f : \mathbb{R} \to M_3$ is given by

$$f(x) = I_3.$$

That is, f is the constant function that is I_3 everywhere. Since $M_3 \subset \mathcal{M}$, we must have that f is a matrix function, by definition.

Exercise 81. Find an example of a matrix function f that is not constant. Show that f is a matrix function.

Example 15.26. $A : \mathbb{R} \to \mathbb{V}_3$ given by

$$A(t) = \begin{bmatrix} t^3 \\ t^2 \\ t^3 \end{bmatrix}$$

is a matrix function, by definition, since $\text{Range}(A) \subset \mathbb{V}_3 = M_{3\times1}$.

Definition 15.27 (Trace of a Complex Matrix)**.** If $A = (a_{i,j})_{i,j=1}^{m,n} \in M_{m\times n}(\mathbb{C})$, then we define the *trace of* A by

$$\text{Tr}(A) = \sum_{i=1}^{m} a_{i,i}.$$

Example 15.28. The reader can see that by definition of trace above,

$$Tr\left(\begin{bmatrix} 9 & -4 & 0 \\ -10 & 205 & -11 \end{bmatrix} \right) = 9 + 205 = 224.$$

Exercise 82. Find $\text{tr}(I_4)$.

Exercise 83. Prove that
(a) $\text{tr}(I_n) = n \ \forall n \in \mathbb{N}$ and that
(b) $\text{tr}(0_{m\times n}) = 0 \ \forall m, n \in \mathbb{N}$.

16. Matrix algebra(s)

Definition 16.1 (Complex Matrix Operations)**.** Suppose that $A, B \in M_{m\times n}(\mathbb{C})$ and that $z \in \mathbb{C}$. Then we define

(a) $\overline{A}$ to be $(\overline{a_{i,j}})_{i,j}^{m,n}$.

(b) $A + B$ to be $(a_{i,j} + b_{i,j})_{i,j=1}^{m,n}$,

(c) $-A$ to be $(-a_{i,j})_{i,j=1}^{m,n}$,

(d) zA to be $(za_{i,j})_{i,j=1}^{m,n}$,

(e) $A - B$ to be $(a_{i,j} + b_{i,j})_{i,j=1}^{m,n}$.

Let $C \in M_{r\times s}(\mathbb{C})$. Then

(f) the *tensor product* of A and C is denoted by $A \otimes C$ and is defined to be $(a_{i,j}C)_{i,j=1}^{m,n}$.

Let $D \in M_{n\times p}(\mathbb{C})$. Then

(g) the *matrix product* AD of A and D is the matrix P, where

$$p_{i,j} = \sum_{k=1}^{n} a_{i,k} d_{k,j}$$

$\forall i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, p\}$.

Let $E, F \in M_n(\mathbb{C})$. Then we

(h) define the *bracket product* or Lie bracket of E and F by

$$[E, F] := EF - FE,$$

and

(i) the *circle product* of E and F by

$$E \odot F := \frac{1}{2}(EF + FE).$$

Let $U, V \in M_{m\times n}(\mathbb{C})$. Then we define the *direct product* of U and V by

$$U \times_d V = (u_{i,j} v_{i,j})_{i,j=1}^{m,n}.$$

Remark: Although the direct product of two matrices of the same size is a very natural and simple operation, it is surprisingly not as important or useful as the other types of matrix products! Matrix products of the other types turn out to be very important in theoretical (and applied) physics.

All of the above operations are straightforward generalizations of the real matrix operations that the reader has already seen in precalculus. Thus we strongly encourage the reader to complete the following exercises:

Exercise 84. Find $2I_2 - 5I_2$, $-I_3$, $I_3 - I_3$, and $I_4 I_4$.

Exercise 85. Find $\overline{[1+6i \;\; 7 \;\; -4-58i \;\; i-7]}$.

Exercise 86. Show that $\overline{A} = A$ if $A \in M_{m\times n}$.

Exercise 87. Find $[A, B]$ if $A = [7+i]$ and $B = [-10-2i]$.

Exercise 88. (a) Find $A \odot B$ if A and B are given as in the last exercise.
(b) Choose two matrices $C, D \in M_3(\mathbb{C})$.
(c) Compute $C \odot D$.
(d) Compute $[C, D]$.
(e) Find CD.
(f) Find $C \times_d D$.
(g) Find $C \otimes D$.

Exercise 89. Suppose that

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 5 & -9 \\ 13 & -17 & 21 \\ -25 & 29 & -33 \end{bmatrix}.$$

Repeat parts (b)-(g) in the above exercise, for these two matrices C and D.

Theorem 16.2 (Algebraic Properties of Matrices). *Suppose that $A, B, C \in M_{m\times n}(\mathbb{C})$ and that $z, w \in \mathbb{C}$. Then we have that*
(a) $A + B = B + A$,
(b) $A + (B + C) = (A + B) + C$,
(c) $1A = A$,
(d) $A + 0_{m\times n} = A$,
(e) $A + (-A) = 0_{m\times n}$,
(f) $0A = 0_{m\times n}$, and
(g) $(zw)A = z(wA)$.
Furthermore, let $D \in M_{n\times p}(\mathbb{C})$. Then we have that
(h) $(A + C)D = AD + CD$,
If $E \in M_{p\times q}(\mathbb{C})$, then
(i) $(AD)E = A(DE)$.

Exercise 90. Prove all of these properties, which are straightforward generalizations of what you already knew about real matrices before enrolling in this course.

Exercise 91. Which properties (a),(b), (d), and/or (e) above hold if "+" is removed? Prove your answer.

Exercise 92. Which properties (a),(b), (d) above hold if "+" is replaced by
(i) "$\times_d$"?

(ii) "$\otimes$"?
(ii) "$\odot$"?
As usual, you should prove your answers.

Exercise 93. Is it true that if $A, B \in M_n(\mathbb{C})$, then $[A, B] = [B, A]$? If the answer is yes, then prove it. Otherwise, give an example of two choices of A and B for which $[A, B] \neq [B, A]$.

Exercise 94. Determine whether or not the following equations hold, and prove your answers:
(i) $A \times_d (B + C) = A \times_d B + A \times_d C$ $\forall m, n \in \mathbb{N}$ and $A, B, C \in M_{m\times n}(\mathbb{C})$.
(ii) $A \otimes (B + C) = A \otimes B + A \otimes C$ $\forall m, n, p \in \mathbb{N}$, $A \in M_{m\times n}(\mathbb{C})$, and $B, C \in M_{n\times p}(\mathbb{C})$.
(iii) $A \odot (B + C) = A \odot B + A \odot C$ $\forall n \in \mathbb{N}$ and $A, B, C \in M_n(\mathbb{C})$.
(iv) $A \times_d (B \times_d C) = (A \times_d B) \times_d C$ $\forall m, n\mathbb{N}$ and $A, B, C \in M_{m\times n}(\mathbb{C})$.
(v) $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ $\forall m, n, p, q, r \in \mathbb{N}$ and $A \in M_{m\times n}(\mathbb{C})$, $B \in M_{n\times p}$, and $C \in M_{q\times r}(\mathbb{C})$.
(vi) $A \odot (B \odot C) = (A \odot B) \odot C$ $\forall m, n \in \mathbb{N}$ and $A, B, C \in M_{m\times n}(\mathbb{C})$.
(vii) $[[A, B], C] = [A, [B, C]]$ $\forall n \in \mathbb{N}$ and $A, B, C \in M_n(\mathbb{C})$.

Theorem 16.3 (Matrix Times a Column Theorem)**.** Let $A = [\vec{a}_1|\vec{a}_2| \ldots, |\vec{a}_n] \in M_{m\times n}(\mathbb{C})$,where $\vec{a}_i \in \mathbb{V}_n^{\mathbb{C}}$ $\forall i \in \{1, 2, \ldots, n\}$. Let $\vec{c} \in \mathbb{V}_n^{\mathbb{C}}$. Then,

$$A\vec{c} = \sum_{i=1}^{n} \vec{a}_i c_i$$

Exercise 95. Use the Matrix Times a Column Theorem to compute:

$$\begin{bmatrix} -7 & 6 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} 5 \\ -9 \end{bmatrix}.$$

Exercise 96. Prove the Matrix times a Column Theorem.

Definition 16.4. Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{V}_n$. We call $\vec{x}$ an $\mathbb{R}$-*linear combination* of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ iff $\exists \vec{\alpha} \in \mathbb{V}_m$ such that

$$\vec{x} = \sum_{i=1}^{m} \alpha_i \vec{v}_i$$

.

Example 16.5. Consider: $\vec{e}_1^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then, $\vec{x} = 2\vec{e}_1^{(2)} + 3\vec{e}_2^{(2)}$ is a linear combination of $\vec{e}_1^{(2)}$ and $\vec{e}_2^{(2)}$, by definition of linear combination. Note that here,

$$n = m = 2,\ \ \vec{v}_1 = \vec{e}_1^{(2)}\ \ \vec{v}_2 = \vec{e}_2^{(2)},\ \text{and}\ \alpha = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Definition 16.6. Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{V}_n$. We call $\vec{x}$ a $\mathbb{C}$-*linear combination* of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{V}_n^{\mathbb{C}}$ iff $\exists \vec{\alpha} \in \mathbb{V}_m^{\mathbb{C}}$ such that

$$\vec{x} = \sum_{i=1}^{m} \alpha_i \vec{v}_i$$

.

Exercise 97. Show that $[2 \quad -1 + 7i]^T$ is a $\mathbb{C}$-linear combination of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Remark on vector notation: Up to now, we have been indicating vectors in $\mathbb{V}_n$ with arrows over them, but this turns out to be quite hassle to typeset, so as is common in most textbooks, we will denote vectors with boldface rather than put arrows over them, from now on.

Definition 16.7. If X, Y are sets, then we define

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

Exercise 98. List all of the elements of $\{-2, 0, 4, 57\} \times \{0, e\}$.

In what follows, we will need the following notation:

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The identity matrix I_n can alternatively viewed as the matrix formed by the indicator function of the diagonal of the set $\mathbb{N} \times \mathbb{N}$, as we now explain in more detail below:

Theorem 16.8 (Identity Matrix Kronecker Delta Theorem)**.** $I_n = (\delta_{i,j})_{i,j=1}^n$ where

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \forall i, j \in \{1, 2, \ldots, n\}$$

The theorem's proof is obvious. One might wonder why we have used $\delta_{i,j}$ rather than $i_{i,j}$. A closer look at the latter expression has two occurrences of the letter i that could lead to confusion. Since we then end up choosing a different symbol even though I is our upper case letter used in the expression I_n for the identity matrix of order n, it may be that δ is used here instead of i, for the simple reason that $S \times S$ for a subset S of some set X is called the diagonal of $S \times S$, so that $\delta : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \{0, 1\}$ is a so-called "indicator function" of the diagonal of $S \times S$. Since δ is a Greek lower case d, one might wonder if this δ was chosen at some point, to indicate this diagonal characteristic function δ.

Exercise 99. Find the $(45, 57)$-entry and then the $(3003, 3003)$ entry of I_{9998} using the Identity Matrix Kronecker Delta Theorem.

Theorem 16.9 (Matrix Transpose Theorem)**.** (1) Suppose that $A \in M_{m\times n}(X)$, where X is a non-empty set. Then

$$(A^T)^T = A.$$

(2) If $B \in M_{m\times n}(X)$ and $X = \mathbb{C}$, then

$$(A+B)^T = A^T + B^T.$$

(3) If $\alpha \in \mathbb{C}$ and $A \in M_{m\times n}(\mathbb{C})$, then $(\alpha A)^T = \alpha A^T$.
(4) If $A \in M_{m\times n}(\mathbb{C})$ and $B \in M_{n\times p}(\mathbb{C})$, then

$$(AB)^T = B^T A^T.$$

NOTE: $(AB)^T \neq A^T B^T$, usually. Now, this might disappoint you, but don't let it disappoint you much. It is actually a good thing that this non-commutativity occurs. In other words, the transpose's ability to "see" non-commutativity may indeed be a good thing, not a bad thing. In order to understand how important our remarks here are, the reader is strongly encouraged to complete the following exercise:

Exercise 100. Find two matrices A and B such that the following conditions hold:
(1) The matrix product AB is defined;
(2) $(AB)^T \neq A^T B^T$.
For your chosen matrices, prove that conditions (1) and (2) above hold, rather than simply stating what the matrices are.

Example 16.10. By the first part of the Matrix Transpose Theorem, we know that

$$\left(\begin{bmatrix}1\\2\\3\end{bmatrix}^T\right)^T = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

Exercise 101. Prove the above equation without using the Matrix Transpose Theorem.

Exercise 102. Compute:

$$\left\{\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}^T\right)^T\right\}^T,$$

(a) by use of the definition of the transpose;
(b) by use of the Matrix Transpose Theorem.

Example 16.11. By the second part of the Matrix Transpose Theorem, we obtain that

$$\begin{aligned}(I_2+I_2)^T &= I_2^T+I_2^T\\ &= I_2+I_2\\ &= 1I_2+1I_2\\ &= (1+1)I_2\\ &= 2I_2\\ &= 2\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ &= \begin{bmatrix}2(1)&2(0)\\2(0)&2(1)\end{bmatrix}\\ &= \begin{bmatrix}2&0\\0&2\end{bmatrix}.\end{aligned}$$

Notice that the second line above follows from Example 15.7.

Exercise 103. Compute

$$\left(\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}+\begin{bmatrix}2&3&4\\5&6&7\end{bmatrix}\right)^T$$

by

(a) using the second part of the Matrix Transpose Theorem;
(b) not using the second part of the Matrix Transpose Theorem.

Example 16.12. By the third part of the Matrix Transpose Theorem, we have that

$$(-3I_2)^T=-3I_2^T=-3I_2=-3\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}-3(1)&-3(0)\\-3(0)&-3(1)\end{bmatrix}=\begin{bmatrix}-3&0\\0&-3\end{bmatrix}.$$

Exercise 104. Compute

$$\left(50\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\right)^T$$

by

(a) using the third part of the Matrix Transpose Theorem;
(b) without using the Matrix Transpose Theorem.

Exercise 105. Suppose that $A, B \in M_{m\times n}(\mathbb{C})$, and let $\alpha, \beta \in \mathbb{C}$.

(1) Show that $(-A)^T = -A^T$.
(2) Show that

$$(\alpha A+\beta B)^T=\alpha A^T+\beta B^T.$$

(3) Show that $(A-B)^T = A^T - B^T$.

Remark: One might wonder if the transpose is a linear operator. One must be careful about answering this question, as the following exercise will show anyone who completes it successfully.

Exercise 106. Let $T : \mathcal{M} \to \mathcal{M}$ be given by $T(A) = A^T$. (T is called the *transpose operator.*)

(a) Find

$$T(\begin{bmatrix} \text{Joe} & 9 & \text{Dana} \end{bmatrix}).$$

(b) Is the set of all complex matrices a vector space with standard matrix addition and scalar matrix multiplication? Prove your answer.

(c) Is the transpose operator a linear operator from the collection of all complex matrices to itself? Prove your answer.

(c) Show that $\forall n \in \mathbb{N}$, $T|_{M_n(\mathbb{C})}$, the restriction of the transpose operator to $M_n(\mathbb{C})$ is indeed a linear operator on $(M_n(\mathbb{C}), +, \cdot)$ with usual matrix entry-wise addition and scalar matrix multiplication.

Example 16.13. The fourth part of the Matrix Transpose Theorem implies that

$$\begin{aligned} \left\{ \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right)^2 \right\}^T &= \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right)^T \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T. \end{aligned}$$

We used the theorem in the second line above. Now by definition of transpose, we have that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = (b_{i,j})_{i,j=1}^2,$$

where $b_{i,j} = a_{j,i}$ for $i, j = 1, 2$ and

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (a_{i,j})_{i,j=1}^2.$$

Thus we have that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \left\{ \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right)^2 \right\}^T &= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 3(2) & 1(3) + 3(4) \\ 2(1) + 4(2) & 2(3) + 4(4) \end{bmatrix} \\ &= \begin{bmatrix} 1 + 6 & 3 + 12 \\ 2 + 8 & 6 + 16 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}. \end{aligned}$$

Research Project: Formulate analogues of the third part of the Matrix Transpose Theorem for

(a) the bracket product,
(b) the circle product,
(c) the direct product, of two matrices,
(d) the tensor product of two matrices.

Determine which of these three analogues is true or false. If the result is true, prove it, and if it is not true, give a counterexample showing that the statement is false. If you get any results on this question, share it with us at PSOMS!

Exercise 107. Compute

$$\left(\begin{bmatrix}1 & 3\end{bmatrix}^T \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)^T$$

two ways:
(a) by using the fourth part of the Matrix Transpose Theorem;
(b) by not using the fourth part of the Matrix Transpose Theorem.

Theorem 16.14 (Triangular Matrix Product Invariance Theorem)**.** Suppose that $A, B \in M_{m\times n}(\mathbb{C})$. Then the following statements hold:
(1) If A and B are lower triangular, then AB is as well.
(2) If A and B are upper triangular, then AB is as well.
(3) If A and B are unit lower triangular, then AB is as well.
(4) If A and B are unit upper triangular, then AB is as well.
(5) If A and B are diagonal, then AB is as well.

Example 16.15. Let $A = B = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I_2$

The theorem says that $AB = (I_2)^2$ is upper triangular, lower triangular, unit upper triangular, unit lower triangular, and diagonal, since we showed in Example 15.14 that I_2 is all of these.

Exercise 108. Determine whether or not

$$\begin{bmatrix}1 & 0\\ 0 & -7\end{bmatrix}\begin{bmatrix}29 & 0\\ 1 & -1\end{bmatrix}$$

is
(a) lower triangular
(b) unit lower triangular
(c) upper triangular
(d) unit upper triangular
(e) diagonal, by
(i) using the Triangular Matrix Product Invariance Theorem;
(ii) by not using that theorem.

Exercise 109. Prove the Triangular Matrix Product Invariance Theorem.

Research Project: Which of the statements in the theorem hold if AB is replaced by
(a) $[A, B]$;
(b) $A \times_d B$;

(c) $A \otimes B$;
(d) $A \odot B$?

Prove your answers, or give counterexamples and show why your counterexamples are indeed counterexamples, in any such case. If you obtain any results on this, come share your results with us at PSOMS or at a Math Colloquium!

Definition 16.16. Let $L \in M_{m\times n}$, and let $a \in \mathbb{R}$. Suppose that $\forall i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, we have that $a_{i,j}$ is a real-valued function. We say that

$$\lim_{x\to a} A(x) = L$$

iff $\forall i \in \{1, 2, \ldots, m\}$ and $\forall j \in \{1, 2, \ldots, n\}$,

$$\lim_{x\to a} a_{i,j}(x) = L_{i,j}$$

.

Since the above definition is a straightforward generalization of the limit of a real-valued function at a real number, we will not elaborate on the above definition and instead ask that the reader complete the following exercise:

Exercise 110. Find

$$\lim_{x\to 0} \begin{bmatrix} \frac{\sin x}{x} & e^x \\ \cos x & x^3 + 10 \end{bmatrix},$$

if this limit exists, and prove your answer.

Exercise 111. Attempt to formulate the definition of a left-sided matrix function limit at a real number (respectively, find a formulation of the definition of a right-sided matrix function limit at a real number). Give examples to illustrate that your formulations are reasonable. Then formulate and prove a Two-Sided Matrix Function Limit Theorem and give an example that applies the theorem.

Definition 16.17. Suppose that A is an $m \times n$ complex matrix function, and let $x \in \mathbb{R}$. We say that A is differentiable at $x \in \mathbb{R}$ iff $\forall i \in \{1, 2, \ldots, m\}$ and $\forall j \in \{1, 2, \ldots, n\}$ and $a_{i,j}$ is differentiable at x . In this case, we define $A'(x)$ $\forall x \in D$ by

$$A'(x) = (a'_{i,j}(x))_{i,j=1}^{m,n}.$$

Again, the above definition is an easy generalization of the notion of a one-variable derivative from first-semester calculus. Nevertheless, the reader should complete the following exercise:

Exercise 112. Prove that $f : \mathbb{R} \to M_4$ given by

$$f(x) = I_4$$

is differentiable at all $x \in \mathbb{R}$, and find the value of $f'(x)$ at all such x.

Definition 16.18. Suppose that A is an $m \times n$ matrix function whose components are all real-valued. We say that $\mathcal{A}$ is an *antiderivative of* A *on Dom* (A) iff $\mathcal{A} : Dom(A) \to M_{m\times n}$, $\mathcal{A}$ is differentiable at all $x \in \text{Dom}(A)$, and, for all such x, we have that $\mathcal{A}'(x) = A(x)$.

Again, since the above definition is a very straightforward generalization of the notion of real-valued antiderivative that you encountered in first semester calculus, we will not delve into further discussion of this definition; however, the reader is strongly encouraged to complete the following exercise:

Exercise 113. Find an antiderivative of $A : \mathbb{R} \to M_2$ given by

$$A(x) = \begin{bmatrix} x^2 & e^{3x+7} \\ \cos 4x & \frac{1}{\sqrt{x^2-10}} \end{bmatrix}.$$

Definition 16.19. Suppose that $A : D \to M_{m\times n}$. We denote by $\int A(x)dx$ the set of all antiderivatives of A, and we call this expression the *indefinite integral of A with respect to the real (dummy) variable x.*

Once again, the above notion is a straightforward generalization of the indefinite integral of a real-valued function, so we will discontinue further discussion of the notation and instead encourage the reader to try the following exercise:

Exercise 114. Compute $\int I_7 \, dx$.

Definition 16.20. Suppose that $a < b$ and that $A : [a, b] \to M_{m\times n}$. We call A Riemann integrable iff $a_{i,j} : [a, b] \to \mathbb{R}$ is Riemann integrable $\forall i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. In this case, we define

$$\int_a^b A(x)dx := \left(\int_a^b a_{i,j}(x)dx \right)_{i,j=1}^{m,n}.$$

Since the above definition is again a straightforward generalization of what the reader is already familar in the case that $m = n = 1$, for example, then we will not delve into further discussion above the above definition. Instead, here is an exercise, which the reader is strongly advised to complete:

Exercise 115. Find

$$\int_0^2 \begin{bmatrix} \sin t & \cos t \\ e^{3t} & (5x-6)^{100} \end{bmatrix} dx.$$

Exercise 116. If A and B are differentiable at $x \in \mathbb{R}$, where $A : D \to M_{mxn}$ and $B : D \to M_{nxp}$ for some $D \subset \mathbb{R}$, then show that the matrix product rule holds:

$$\frac{d}{dx}(AB)(x) = A(x)B'(x) + A'(x)B(x).$$

Research Project: Is there a product rule for the bracket product, the circle product, or the direct product? Prove your answers. If you are able to answer any of these questions, come to PSOMS or the Math Colloquium and give a talk about your results obtained.

17. Systems of Linear Equations

The following definition slightly generalizes the familiar real system of linear equations in one or several real variables that is taught in US high schools. We only generalize it to the case where the variables and coefficients are complex numbers. This generalization immediately applies to a number of things, which we will be brief about if the reader would have already seen the given item in the real case.

Definition 17.1. A complex system of linear equations is a collection of one or more equations, together which can be written in the form

$$\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \ldots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \ldots + a_{m,n}x_n &= b_m,
\end{aligned}$$

where $a_{i,j} \in \mathbb{C}\ \forall i \in \{1, 2, \ldots, m\}$ and $\forall j \in \{1, 2, \ldots, n\}$ and $\underbrace{x_1, x_2, \ldots, x_n}_{x_i} \in \mathbb{C}$ are unknown.

The $a_{i,j}$'s are called the *system coefficients* and the x_i's are called the *system unknowns*. The b_i's are called the *system constants*.

Example 17.2.

$$2x - 3y = 7$$
$$5x + 6y = 9$$

is a system of linear equations.
The system coefficients are the entries of: $\begin{bmatrix} 2 & -3 \\ 5 & 6 \end{bmatrix}$.
The system unknowns are $x_1 = x$ and $x_2 = y$.
The system constants are 7 and 9.

Exercise 117. Give an example of a complex linear system of three equations in three complex unknowns x, y, and z that is not a real system of linear equations. Prove that your answer is correct.

Definition 17.3. A complex system of linear equations is called

(1) *homogeneous* iff all of its system constants are 0 and
(2) *nonhomogeneous* iff it's not homogeneous.

Example 17.4. The system of linear equations in the last example is not homogeneous because the system coefficients are 7 and 9 rather than 0.

Example 17.5. The system of linear equations

$$\begin{aligned}
3x - y + z &= 0 \\
2x + y - z - 5 &= -5 \\
x + y + z &= z
\end{aligned}$$

is homogeneous by definition, because it can be written as

$$\begin{aligned}
3x - y + z &= 0 \\
2x + y - z &= 0 \\
x + y &= 0,
\end{aligned}$$

whose system constants are all 0.

Exercise 118. Give an example of a non-real system of linear equations that is
(a) homogeneous
(b) nonhomogeneous.

Definition 17.6. We say that $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$ is a solution to a complex system of linear equations with system unknowns $x_1, x_2, \ldots, x_n$ iff all equations in the system are true when $\forall i \in \{1, 2, \ldots n\}$, x_i is replaced by c_i.

Remark: Again, the reader already saw the above definition in the special case where the system is a real one and the unknowns are real. Thus we will again be brief about discussing the above definition any further.

Exercise 119. Give an example of a complex, non-real solution to a non-real, complex linear system of equations in at least two complex unknowns.

Definition 17.7. We say that a system of complex linear equations in n complex unknowns is consistent iff it has at least one solution. Systems of complex linear equations that are not consistent are called inconsistent.

Once again, the above definition is known to students in this course in the special case of real equations and real unknowns, so the above generalization is not one that will cause difficulty. Thus we will be brief in our discussion of this terminology.

Exercise 120. Give an example of a
(1) consistent, non-real, complex linear system of equations in at least two complex variables; and
(2) inconsistent, non-real complex linear system of equations in at least two complex variables.

Definition 17.8. Suppose that a complex system of linear equations is written in the form

$$\begin{gathered}
a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n}x_n = b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \ldots + a_{2,n}x_n = b_2 \\
\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \ldots + a_{m,n}x_n = b_m.
\end{gathered}$$

Then the matrix $A = (a_{i,j})_{i=1,j=1}^{mn}$ is called the coefficient matrix of the system.

Example 17.9.

$$\begin{aligned}
2x - 5y &= -9 \\
5x + 7y &= 16
\end{aligned}$$

has coefficient matrix $\begin{bmatrix} 2 & -5 \\ 5 & 7 \end{bmatrix}$, by definition.

Exercise 121. For the examples you gave in Exercise 120, give their coefficient matrices.

Theorem 17.10 (Linear System Matrix Equation Theorem)**.** *A complex system of linear equations with coefficient matrix A, system unknowns $x = (x_1, x_2, \ldots, x_n)$, and system constants $b = (b_1, b_2, \ldots, b_n)$ is true iff*

$$A\vec{x} = \vec{b}$$

Example 17.11. The Linear System Matrix Equation Theorem guarantees that the system of linear equations in Example 17.9 is true iff the following matrix equation holds:

$$\begin{bmatrix} 2 & -5 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ 16 \end{bmatrix}.$$

Exercise 122. Find a matrix equation that is true iff $-4x_1 + 8x_2 - ix_3 + x_4 = 7$.

Exercise 123. Prove the Linear System Matrix Equation Theorem.

Definition 17.12. Suppose that a complex system of linear equations has coefficient matrix $A \in M_{m\times n}(\mathbb{C})$, unknown vector $\vec{x} \in \mathbb{V}_n^{\mathbb{C}}$, and system coefficient vector $\vec{b} \in \mathbb{V}_m^{\mathbb{C}}$. The equation $A\vec{x} = \vec{b}$ is sometimes also called the matrix-vector form of the system.

Definition 17.13. The matrix

$$\left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \dots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} & b_m \end{array}\right]$$

is called the augmented matrix form of the system of complex linear equations given in Definition 19.11 above. We sometimes more briefly denote this augmented matrix by

$$A|\vec{b}$$

Example 17.14. $\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 5 & 6 & -3 \end{array}\right]$ is the augmented matrix of the system of linear equations with matrix-vector form

$$\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

which is the matrix-vector form of

$$2x + 3y = 1$$

$$5x + 6y = -3.$$

Exercise 124. Find the
(a) augmented matrix; and
(b) matrix equation of

$$\begin{aligned} (1+i)x - iy + (-7+3i)z &= 0 \\ 6x - iy + (5-i)y + z &= -1 \\ -ix + 2y - (3-i)z &= 3, \end{aligned}$$

assuming that $x, y, z \in \mathbb{C}$ are unknowns.

Theorem 17.15 (Vector-Matrix System of ODE's Theorem)**.** *Suppose that $a_{i,j} : I \to \mathbb{R}$ $\forall i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ and $b_k : I \to \mathbb{R}$ $\forall k \in \{1, 2, \ldots, m\}$, where I is a nondegenerate interval. Then we have that the following system of ode's*

$$\begin{array}{cccccc} \frac{dx_1}{dt} = & a_{1,1}x_1 & +a_{1,2}x_2 & +\ldots & +a_{1,n}x_n & +b_1 \\ \frac{dx_2}{dt} = & a_{2,1}x_1 & +a_{2,2}x_2 & +\ldots & +a_{2,n}x_n & +b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{dx_m}{dt} = & a_{m,1}x_1 & +a_{m,2}x_2 & +\ldots & +a_{m,n}x_n & +b_m, \end{array}$$

in unknowns $x_1, x_2, \ldots, x_n$ holds iff, letting $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $A = (a_{i,j})_{i=1,j=1}^{mn}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, we have that the equation $\vec{x}'(t) = A(t)\vec{x} + \vec{b}(t)$ holds. (This equation is called the <u>vector-matrix form</u> of the above system of ode's).

Example 17.16. Consider $x_1' = (\sin(t))x_1 + x_2 + 100$ and $x_2' = -7\pi x_1 - (\tan(t))x_2 - \sec(t)$. Where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are unknown differentiable functions. Then the theorem says that the system is the same as

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \sin(t) & 1 \\ -7\pi & -\tan(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 100 \\ -\sec(t) \end{bmatrix}$$

Exercise 125. Find the vector-matrix form of the following system of ode's:

$$\begin{aligned} x_1' &= \cos\left(t + \frac{\pi}{4}\right) x_1 - (t^2 - 5)x_2 - x_3 + \pi \\ x_2' &= -7x_1 + \ln(t)x_2 + 6x_3 - 1 \\ x_3' &= 58x_1 - x_2 + (\tan t)x_3. \end{aligned}$$

Exercise 126. Prove the Vector-Matrix System of ODE's Theorem.

18. Elementary Row Operations on Complex Matrices, Row-Echelon Form Complex Matrices, and Reduced Row-Echelon Form Complex Matrices

With the exception of rank and rank estimation for matrices, everything that we will discuss in this section simply extends what the reader already knows about elementary row operations, row-echelon form, and reduced row echelon form matrices from the case of real matrices to the case of complex matrices. The definitions are no different in the complex case, so rather than spend space on examples, which the reader should already be aware of in the real case via precalculus/college algebra, we strongly encourage the reader to try the exercises that accompany each item. Later in this section, we will define the rank of a complex matrix, and this will be the only new material for the reader in addition to other facts about rank of a matrix. Thus we will be more expansive about examples and exercises when treating that material toward the end of the section.

Definition 18.1. $A \in M_{m\times n}(\mathbb{C})$ is said to be in row echelon form iff both of the following conditions hold:

(1) If $i \in \{1, 2, \dots m\}$, $a_{i,j} = 0\ \forall j \in \{1, 2, \dots, n\}$ and $a_{k,l} \neq 0$ for some $k \in \{1, 2, \dots, m\}$ and $l \in \{1, 2, \dots, n\}$, then $k < i$.
(2) If $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$, $a_{i,j} \neq 0$, and $a_{i,l} = 0\ \forall l \in \{1, 2, \dots, j-1\}$, then $a_{i,j} = 0$.
(3) If $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots n\}$, $a_{i,j} = 1$, with $a_{i,l} = 0\ \forall l \in \{1, 2, \dots, j-1\}$, $p > i$, and $a_{p,l} = 1$, with $a_{p,q} = 0$ for all $q \in \{1, 2, \dots, l-1\}$, then $q > j$.

Exercise 127. Prove that I_2 and I_5 are in row-echelon form, by definition. Find an example of a non-real matrix that is in row echelon form, and prove your answer. Finally, find a non-real matrix that is not in row-echelon form, and prove your answer.

Exercise 128. Suppose that $A \in M_{m\times n}(\mathbb{C})$ and that A is in row-echelon form. Must A be upper triangular? Prove your answer.

Exercise 129. Suppose that $A \in M_{m\times n}(\mathbb{C})$ and that A is upper triangular. Is A in row echelon form? Prove your answer.

Remark: The definition of row-echelon form is intimidating-lookng, but we can assign intuitive phrases to each of the conditions in the definition so that the concept makes sense while working with it, as follows:

Definition 18.2 (non-zero row of a complex matrix)**.** If $i \in \{1, 2, \dots, m\}$ and $A \in M_{m\times n}(\mathbb{C})$, where $m, n \in \mathbb{N}$, we say that the ith row of A is *non-zero* iff $a_{i,j} \neq 0$ for some $j \in \{1, 2, \dots, n\}$.

Exercise 130. Show that both rows of I_2 are non-zero. Find an example of an non-real matrix that contains at leat one non-zero row, but having at least one row that is not non-zero. Prove your answer.

Definition 18.3 (leading coefficient of a non-zero row in a complex matrix)**.** Suppose that $A \in M_{m\times n}(\mathbb{C})$, where $m, n \in \mathbb{C}$, and suppose that Row i of A is non-zero. If $j \in \{1, 2, \dots, n\}$, then we call $a_{i,j}$ the *leading entry* of Row i iff $a_{i,l} = 0\ \forall l \in \{1, 2, \dots, j-1\}$ and $a_{i,j} \neq 0$.

Exercise 131. Find the leading entry in each non-zero row of I_3, if there any such rows. Are there any leading entries in I_2 that are not 1's? Can you find an example of a complex matrix with nothing but non-real entries and no leading coefficients that are not in the first column?

Definition 18.4 (To the right in a matrix)**.** Suppose that $m, n \in \mathbb{N}$, $A \in M_{m\times n}(\mathbb{C})$, $i, k \in \{1, 2, \dots, m\}$, and $j \in \{1, 2, \dots, n\}$. We say that $a_{i,j}$ is *to the right of* $a_{k,l}$ iff $j > k$.

Exercise 132. Show that the entry $-2 + 7i$ is to the right of the entry 3 in

$$\begin{bmatrix} 1 & -2+7i \\ 3 & 4. \end{bmatrix}$$

Show that the entry $-2 + 7i$ is *not* to the right of the entry 4 in the matrix, nor to the right of itself.

The above definitions lead to the following friendlier-looking characterization of row-echelon form complex matrices:

Theorem 18.5 (Row Echelon Form Characterization Theorem)**.** *$A \in M_{m\times n}(\mathbb{C})$ is in row-echelon form iff all of the following conditions are met:*

1. *If there are any all-zero rows in A, they are not above any non-zero rows in A.*
2. *The leading coefficient of any non-zero row of A is a 1.*
3. *The leading coefficient of any row is to the right of any leading coefficient in a higher row.*

Exercise 133. Show that I_4 is in row echelon form using the Row Echelon Form Matrix Characterization Theorem.

Exercise 134. Show that

$$\begin{bmatrix} 1 & 0 & 0 & 9-i \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row-echelon form, using the Row-Echelon Form Characterization Theorem.

Exercise 135. By replacing exactly one entry in the above matrix, show that the matrix that you thus obtain is not in row-echelon form. Prove your answer using the Row-Echelon Characterization Theorem.

Definition 18.6 (Leading 1 in a Row-Echelon Form Matrix)**.** The leading entry of any non-zero row in a row echelon form matrix $A \in M_{\times n}(\mathbb{C})$ is called a "leading 1" of A.

Exercise 136. Find an example of a non-real matrix in $M_{4\times 5}(\mathbb{C})$ that is in row-echelon form, with 2 leading 1's. Prove your answer.

Definition 18.7. $A \in M_{m\times n}(\mathbb{C})$ is said to be in reduced row echelon form iff

1. A is in row echelon form.
2. If $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, and the (i, j)-entry of A is a leading 1, then $a_{i,k} = 0\ \forall k \in \{1, 2, \ldots j-1, j+1, j+2, \ldots m\}$.

Exercise 137. Find an example of an $A \in M_{3\times 5}(\mathbb{C})$ with at least one non-real entry and such that A is in reduced row echelon form and has neither 1's nor 0's in the fifth column. Prove that A satisfies these conditions by definition, of course.

Theorem 18.8 (Reduced Row-Echelon Form Characterization Theorem)**.** Suppose that $m, n \in \mathbb{N}$ and that $A \in M_{m\times n}(\mathbb{C})$. Then A is in reduced row echelon form iff A is in row-echelon form and all other entries in the same column as any leading 1 in A are 0.

Exercise 138. Prove that the example you found above is in reduced row echelon form, using the Reduced Row-Echelon Form Characterization Theorem.

Definition 18.9. The following operations on a matrix $A \in M_{mxn}(\mathbb{C})$ are called the *elementary row operations* on A:

1. Multiply any row of A by a non-zero complex number.
2. Interchange any 2 rows of A.

(3) Multiply any row of A by a non-zero complex number, add the resulting row matrix to any row in A and place that row by the resulting n-tuple (row matrix).

We write the above elementary row operations on A in more precise algorithmic form as follows:

(1) If $k \in \{1, 2, \ldots, m\}$, and $r \in \mathbb{C}$ is not 0, then to *multiply the kth row of A by r* means to write the matrix $B \in M_{m \times n}(\mathbb{C})$, where

$$b_{i,j} = a_{i,j} \;\; \forall i \in \{1, 2, \ldots, k-1, k+1, k+2, \ldots, m\} \text{ and } b_{k,j} = ra_{k,j} \;\; \forall j \in \{1, 2, \ldots, n\}.$$

(2) If $k, l \in \{1, 2, \ldots, m\}$, then to *interchange the kth and lth rows of A* means to write down the matrix C, where $c_{i,j} = a_{i,j}$ $\forall i \in \{1, 2, \ldots, m\}$ such that $i \neq k, l$, and $\forall j \in \{1, 2 \ldots, n\}$, and for all such j, $c_{k,j} = a_{l,j}$ and $c_{l,j} = a_{k,j}$.

(3) If $k, l \in \{1, 2, \ldots, m\}$ and $r \in \mathbb{C}$, then to *multiply the kth row of A by r, add this row to row l of A, and replace row l of A by the result*, means to write down the matrix D, where $d_{i,j} = a_{i,j}$ $\forall i \in \{1, 2, \ldots, l-1, l+1, l+2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ and, $\forall$ such j, $d_{l,j} = ra_{k,j} + a_{l,j}$.

Exercise 139. Show that the following facts hold about $A = \begin{bmatrix} i & 2 \\ 3 & 4 \end{bmatrix}$.

(a) $\begin{bmatrix} 3 & 4 \\ i & 2 \end{bmatrix}$ can be obtained from A by interchanging the first and second rows of A.

(b) $\begin{bmatrix} 1 & -2i \\ 3 & 4 \end{bmatrix}$ can be obtained from A by multiplying the first row of A by $-i$.

(c) $\begin{bmatrix} i & 2 \\ 4 & 4-2i \end{bmatrix}$ can be obtained from A multiplying the first row of A by $-i$, adding the resulting row to the second row of A, and replacing the second row of A by the resulting row.

Exercise 140. Let $A \in M_{m \times n}(\mathbb{C})$. Show that the following facts hold:

(a) $\forall r \in \mathbb{C}$ such that $r \neq 0$ and $i \in \{1, 2, \ldots, m\}$, $\exists! E^{\times r, i, m} \in M_m(\mathbb{C})$ such that $E^{\times r, i, m} A$ is the matrix obtained by multiplying the ith row of A by r. *Hint:* Consider the matrix obtained from I_m by multiplying the ith row of I_m by r.

(b) $\forall k, l \in \{1, 2 \ldots, m\}$, $\exists! E^{k \leftrightarrow l, m} \in M_m(\mathbb{C})$ such that $E^{k \leftrightarrow l, m} A$ is the matrix obtained by interchanging rows k and l in A.

(c) $\forall k, l \in \{1, 2, \ldots, m\}$, $\exists! E^{\times r, k \to l, m} \in M_m(\mathbb{C})$ such that $E^{\times r, k \to l, m} A$ is the matrix by replacing the lth row of A by the row obtained by multiplying the kth row of A by r and adding the resulting row to row l of A.
(Similar hints apply to parts (b) and (c).)

(d) The matrices described in parts (a)-(c) of this exercise are called the *elementary matrices of order m.* Prove that all of these matrices have non-zero determinant.

(e) Suppose that A and B are elementary matrices of order m? Is AB an elementary matrix of order n?

(f) Repeat (e) for $A \otimes B$, $[A, B]$, $A \circ B$, and $A \times_d B$.

Remark: The matrices $E^{\times r,i,m}$, $E^{k \leftrightarrow l,m}$, and $E^{\times r,k \to l,m}$ are called the *elementary matrices of order m* and are given this name because they are clearly important in any theory and applications of the elementary row operations.

The reader may find it helpful to visualize what happens to a matrix $A = M_{m \times n}(\mathbb{C})$ is when any of the three possible types of elementary row operations are performed on A, as we have pointed out below:

- Let $A = (a_{i,j})_{i,j=1}^{m,n} \in M_{m \times n}(\mathbb{C})$, let $i \in \{1, 2, \ldots, m\}$ and let $r \in \mathbb{C}$ be such that $r \neq 0$. Then multiplying row i of A by r yields the matrix

$$B = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i-1,1} & ra_{i-1,2} & \cdots & a_{i-1,n} \\ ra_{i,1} & ra_{i,2} & \cdots & ra_{i,n} \\ a_{i+i,1} & a_{i+1,2} & \cdots a_{i+1,n} & \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

- Interchanging rows k and l in A, where $k < l$, yields the matrix

$$C = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,n} \\ a_{l,1} & a_{l,2} & \cdots & a_{l,n} \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{l-1,1} & a_{l-1,2} & \cdots & a_{l-1,n} \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \\ a_{l+1,n} & a_{l+1,2} & \cdots & a_{l+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

- Replacing the lth row by the sum of r times the kth row of A and the lth row of A yields the matrix

$$D = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{l-1,1} & a_{l-1,2} & \cdots & a_{l-1,n} \\ ra_{k,1} + a_{l,1} & ra_{k,2} + a_{l,2} & \cdots & ra_{k,n} + a_{l,n} \\ a_{l+1,1} & a_{l+1,2} & \cdots & a_{l+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

Theorem 18.10 (Row Echelon Form Existence Theorem)**.** If $A \in M_{m\times n}(\mathbb{C})$, then there is a finite sequence of elementary row operations that, if performed, yields a row echelon form matrix $B \in M_{m\times n}(\mathbb{C})$.

Example 18.11. Show that I_3 can be made into a row echelon form matrix by interchanging any 2 of its rows twice.

Exercise 141. Show that if

$$A = \begin{bmatrix} i & 2 \\ 3 & 4 \end{bmatrix},$$

then there there is a sequence of elementary row operations that, if performed, transforms A into a matrix that is in row-echelon form, using the Row Echelon Form Existence Theorem. Then, find two different such sequences of elementary operations and show that they yield a matrix that is in row echelon form, but such that the row echelon forms are different from each other, depending on what sequence you pick. (Thus, given a complex matrix A, there is not just one row echelon form matrix that can be obtained from A.)

Exercise 142. Prove the Row Echelon Form Existence Theorem.

Definition 18.12. If $A, B \in M_{m\times n}$, then we say that A and B are row-equivalent and write $A \sim B$ iff B can be obtained from A by a finite sequence of row operations.

Exercise 143. Find a row echelon form matrix B such that $B \sim \begin{bmatrix} i & 2 \\ 3 & 4 \end{bmatrix} = A$.

Theorem 18.13 (Reduced Row-Echelon Form Existence and Uniqueness Theorem)**.** If $A \in M_{m\times n}(\mathbb{C})$, then $\exists! B \in M_{m\times n}(\mathbb{C})$ such that $A \sim B$ and B is in reduced row echelon form.

Example 18.14. The Reduced Row-Echelon Form Existence and Uniqueness Theorem guarantees that $\exists$ a reduced row echelon form $B \in M_2(\mathbb{C})$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim B$.

Exercise 144. Show that there is one and only one $B \in M_2(\mathbb{C})$ such that $A \sim B$ and B is in reduced row-echelon form, where

$$A = \begin{bmatrix} i & 2 \\ 3 & 4 \end{bmatrix}.$$

Indeed, find this matrix B.

Exercise 145. Prove the Reduced Row-Echelon Form Existence and Uniqueness Theorem.

Theorem 18.15 (Row Echelon Form Leading 1 Invariance Theorem)**.** *Suppose that $A, B, C \in M_{m\times n}(\mathbb{C})$, and assume that B, C are in row-echelon form and that $A \sim B, C$. Suppose that $i \in \{1, 2, \ldots, m\}$ and that $j \in \{1, 2, \ldots, n\}$. Then the following facts hold:*

(1) B's (i, j)-entry is a leading 1 iff C's (i, j)-entry is a leading 1.
(2) B has a leading 1 in row i iff C has a leading 1 in row i.
(3) B has a leading 1 in column j iff C has a leading 1 in column j.

Exercise 146. Prove that the two different row-echelon form matrices that you found in Exercise 141 have
(1) leading 1's in the same entries,
(2) leading 1's in the same columns,
(3) leading 1's in the same rows.
Then, verify directly that facts (1), (2), and (3) above hold, without using the Row-Echelon Form Leading 1 Invariance Theorem.

Exercise 147. Prove the Row Echelon Form Leading 1 Invariance Theorem.

The Row Echelon Form Leading 1 Invariance Theorem allows us to make the following definition:

Definition 18.16. Suppose that $A, B \in M_{m\times n}(\mathbb{C})$, $A \sim B$, and that B is in reduced row echelon form. Suppose that $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$, and that the (i, j)-entry of B is a leading 1 of B.
(1) (i, j) is called a *pivot position* in A;
(2) The ith row of A is called a *pivot row* of A;
(3) The jth column of A is called a *pivot column* of A.

Example 18.17. Recall that a row-echelon form (which is actually in reduced row-echelon form) of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Columns 1 and 2 of I_2 contain leading 1's, so columns 1 and 2 are pivot columns of A, by definition. Since both rows of I_2 contain leading 1's, both rows of A are pivot rows, by definition. Finally, since positions $(1, 1)$ and $(2, 2)$ contain leading 1's in I_2, then $(1, 1)$ and $(2, 2)$ are pivot positions of A by definition.

Exercise 148. Suppose that

$$A = \begin{bmatrix} -3+i & -5 & 0 \\ 7-2i & 4+5i & -i \\ 3 & 2 & -1 \end{bmatrix}.$$

Find all of A's
(1) pivot positions;

(2) pivot rows;

(3) pivot columns.

Theorem 18.18 (Row Echelon Non-zero/Zero Rows Number Invariance Theorem)**.** *Let $A \in M_{m\times n}(\mathbb{C})$. Then $\exists! i \in \{1, 2, \dots, m\}$ such that every row-echelon form matrix $B \in M_{m\times n}(\mathbb{C})$ such that $A \sim B$ has i non-zero rows and $m - i$ rows that are all-zero.*

Exercise 149. Let
$$A = \begin{bmatrix} i & 2 \\ 3 & 4 \end{bmatrix}.$$
(a) Find a $B \in M_2(\mathbb{C})$ such that $A \sim B$ and such that B is in row-echelon form. Prove your answer completely.

(b) Show that B has two non-zero rows and no all-zero rows.

(c) Show that if $C \in M_2(\mathbb{C})$ is in row-echelon form and $A \sim C$, then C has two non-zero rows and no all-zero rows.

Since the number of non-zero rows (respectively, all-zero rows) of any row-echelon form matrix B such that $A \sim B$, where $A, B \in M_{m\times n}(\mathbb{C})$, is independent of the choice of B for a given A, by the Row-Echelon Form Non-zero/Zero Rows Invariance Theorem, we can now name this number of non-zero rows, calling it the *rank* of A. More precisely,

Definition 18.19. Let $A \in M_{m\times n}(\mathbb{C})$. We define the rank of A by
$$\text{rank}(A) = \text{the number of non-zero rows in any row echelon matrix } B \text{ that is} \sim A.$$

Example 18.20. A in Example 149 had the property, as we showed then, that any row-echelon form $B \in M_2(\mathbb{C})$ such that $A \sim B$, must have 2 non-zero rows. Hence, by definition, $\text{rank}(A) = 2$.

Exercise 150. Find an example of a non-real 3×5 complex matrix A that has rank 2. Prove your answer.

Theorem 18.21 (Rank Estimation Theorem)**.** *If $A \in M_{m\times n}(\mathbb{C})$, then rank($A$) $\leq m$.*

Example 18.22. Suppose that
$$A = \begin{bmatrix} i & 2 & 3 & -1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{bmatrix}.$$
Then, $A \in M_{3\times 4}(\mathbb{C})$. Therefore, by the Rank Estimation Theorem,
$$\text{rank}(A) \leq 3.$$

Exercise 151. Suppose that
$$A = \begin{bmatrix} i & 2 & 3i & 4 \\ 5 & 6i & 7 & 8i \\ 9i & 10 & 11i & 12 \\ 13 & 14i & 15 & 16i \end{bmatrix}.$$
(a) Without computing $\text{rank}(A)$, find a positive integer m that $\text{rank}(A)$ can be no larger than. Prove your answer.

(b) Find $\text{rank}(A)$.

Exercise 152. Very easily prove the Rank Estimation Theorem.

Exercise 153. Suppose that $A \in M_{m\times n}(\mathbb{C})$. Is $\text{rank}(A) \leq n$? Prove your answer.

19. Gaussian and Gauss-Jordan Elimination for Complex Matrices

Recall from precalculus that Gaussian elimination (respectively, Gauss-Jordan elimination) converts the augmented matrix of a real system of linear equations into a row-echelon (respectively, reduced row-echelon form) matrix which can in turn readily yield the solutions or lack thereof, for the system. One of the things that we point out in this course is that use of the elementary row operations in these two method extends with no changes at all, to the case of complex matrices.

The following definition simply extends the high school mathematics definition of Gaussian elimination for real $m \times n$ matrices:

Definition 19.1 (Gaussian elimination for complex matrices)**.** We call the process of finding a finite sequence of elementary operations that, if performed on $A \in M_{m\times n}(\mathbb{C})$, produces $B \in M_{m\times n}(\mathbb{C})$ in row-echelon form, *Gaussian elimination.*

Exercise 154. Perform Gaussian elimination on the matrix given in Example 18.22.

The following definition simply extends the high school mathematics definition of Gauss-Jordan elimination for real $m \times n$ matrices:

Definition 19.2 (Gauss-Jordan elimination for complex matrices)**.** We call the process of finding a finite sequence of elementary operations that, if performed on $A \in M_{m\times n}(\mathbb{C})$, produces $B \in M_{m\times n}(\mathbb{C})$ in reduced row-echelon form, *Gauss-Jordan elimination.*

Exercise 155. Perform Gauss-Jordan elimination on the matrix given in Example 18.22.

Exercise 156. Suppose that $A \in M_{m\times n}(\mathbb{C})$.

(a) Prove that there is a finite sequence of elementary matrices of order n whose product P satisfies $PA = R$, where $R \in M_{m\times n}(\mathbb{C})$ is in row-echelon form.

(b) Show that the sequence in (a) is not necessarily unique for a given matrix $A \in M_{m\times n}(\mathbb{C})$. That is, give an example of positive integers m and n, such a matrix A, and two different finite sequences of elementary matrices of order m whose respective products P_1 and P_2 satisfy $P_1A = P_2A = B$ for some row-echelon form $B \in M_{m\times n}(\mathbb{C})$.

(c) *Research Project!* Can a statement similar to the statement in (a) be generalized in any way to the other types of matrix products, using either elementary matrices or analogues of them correponding to the various matrix products?

Exercise 157. Suppose that $A \in M_{m\times n}(\mathbb{C})$.

(a) Prove that there is a finite sequence of elementary matrices of order n whose product P satisfies $PA = R$, where $R \in M_{m\times n}(\mathbb{C})$ is in reduced row-echelon form.

(b) Show that the sequence in (a) is not necessarily unique for a given matrix $A \in M_{m\times n}(\mathbb{C})$. That is, give an example of positive integers m and n, such a matrix A, and two different finite sequences of elementary matrices of order m whose respective products P_1 and P_2 satisfy $P_1A = P_2A = B$ for some row-echelon form $B \in M_{m\times n}(\mathbb{C})$.

Theorem 19.3 (Upper Triangular 1's and 0's Main Diagonal Row Echelon Form Theorem)**.** *Let $A \in M_{m\times n}(\mathbb{C})$ be upper triangular, with only 1's and/or 0's on the main diagonal. Then A is in row echelon form.*

Example 19.4. Consider $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Let's prove that this matrix is in row echelon form.

Proof. Let $a_{i,j} =$ the (i,j) entry of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, where $i, j \in \{1, 2\}$. Then we have $a_{1,2} = 0$, so that it is certainly true that $a_{i,j} = 0$ iff $i < j$. Thus we have that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is upper triangular. Also, since $a_{1,1} = a_{2,2} = 0$, we have that the main diagonal of the matrix has only 1's and/or 0's. Hence, by the theorem, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in row echelon form. □

Exercise 158. Use the same theorem to show that some 3×3 non-real complex matrix is in row-echelon form (be sure not to use the definition of row-echelon form here.)

Exercise 159. Prove the theorem.

Theorem 19.5 (Rank Theorem for Complex Matrices)**.** Suppose that $A|\vec{b}$ is the augmented matrix of a system of linear equations with complex coefficients where $A \in M_{m\times n}(\mathbb{C})$ and $\vec{b} \in \mathbb{V}_m^{\mathbb{C}}$. Then, the following facts hold:

(1) If $\text{rank}(A) = \text{rank}(A|\vec{b}) = n$, then $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x} \in \mathbb{V}_n^{\mathbb{C}}$.
(2) If $\text{rank}(A) < \text{rank}(A|\vec{b})$, then $A\vec{x} = \vec{b}$ is inconsistent (has no solution).
(3) If $\text{rank}(A) = \text{rank}(A|\vec{b}) < n$, then the system has infinitely many solutions.

Example 19.6. The system of equations

$$\begin{aligned} x + y &= 2 \\ x - y &= 1 \end{aligned}$$

has augmented matrix $A|\mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Adding -1 times Row 1 of A to Row 2 of A, and replacing Row 2 of A by the resulting row yields the matrix

$$\begin{bmatrix} 1 & 1 \\ -1(1) + 1 & -1(1) + (-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 + 1 & -1 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}.$$

Multiplying Row 2 of the final matrix on the right above by $-1/2$, we obtain the matrix

$$\begin{bmatrix} 1 & 1 \\ \frac{-1}{2}(0) & \frac{-1}{2}(-2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Calling this matrix D, we note that D has $d_{1,1} = d_{2,2} = 1$. Thus the main diagonal of D consists only of 1's and 0's in particular. Also since $i = 2$ and $j = 1$ are the only values in $\{1, 2\}$ for which $i > j$ and $d_{2,1} = 0$, D is upper triangular and has only 1's or 0's on its main diagonal. Thus D is in row-echelon form, by the Upper Triangular/Main Diagonal 1's and 0's Row Echelon Form Theorem. Since $d_{1,1} = 1 = d_{2,2}$ and 1 is not zero, we have that both rows of D are non-zero, by definition of non-zero row. Since D is a row-echelon form matrix that we obtained from A via a finite sequence of elementary row operations, then by definition of rank, A has rank 2.

Next, we compute the rank of the augmented marix $A|\mathbf{b}$. To do so, by definition of rank, we need to find the number of non-zero rows in a row echelon form matrix F that we obtain from $A|\mathbf{b}$ via a finite sequence of elementary row operations. It turns out here that we can use exactly the same sequence of elementary row operations that we performed on A to obtain D above:

Adding -1 times Row 1 of

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right],$$

to Row 2 of $A|\mathbf{b}$, and replacing Row 2 of $A|\mathbf{b}$ by the resulting row yields the matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -1(1)+1 & -1(1)+(-1) & -1(2)+0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ -1+1 & -1-1 & -2+0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -2 \end{array}\right].$$

Multiplying Row 2 of the final matrix on the right above by $-1/2$, we obtain the matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ \frac{-1}{2}(0) & \frac{-1}{2}(-2) & \frac{-1}{2}(-2) \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right].$$

Calling this matrix F, we note that F has $f_{1,1} = f_{2,2} = 1$. Thus the main diagonal of F consists only of 1's and 0's in particular. Also since $i = 2$ and $j = 1$ are the only values in $\{1,2\}$ for which $i > j$ and $d_{2,1} = 0$, F is upper triangular and has only 1's or 0's on its main diagonal. Thus F is in row-echelon form, by the Upper Triangular/Main Diagonal 1's and 0's Row Echelon Form Theorem. Since $F_{1,1} = 1 = F_{2,2}$ and 1 is not zero, we have that both rows of F are non-zero, by definition of non-zero row. Since F is a row-echelon form matrix that we obtained from $A|\mathbf{b}$ via a finite sequence of elementary row operations, then by definition of rank, F has rank 2.

Since $\text{rank}(A) = \text{rank}(A|\mathbf{b})$ here, Part (a) of the Rank Theorem for Complex Systems of Linear Equations implies that our system of linear equations here has exactly one solution.

Exercise 160. Prove that the system of equations in the above example has exactly one solution, without use of the Rank Theorem for Complex Linear Systems of Equations.

Exercise 161. Use the Rank Theorem for Complex Linear Systems of Equations to show that

$$\begin{aligned} x+y+z &= 3 \\ x-y+z &= 1 \\ x+y-z &= 1 \end{aligned}$$

has exactly one solution in $\mathbb{R}^3$.

Exercise 162. Use Gaussian elimination to solve the above system of linear equations.

Example 19.7. The system

$$\begin{aligned} 1x+0y &= 0 \\ 0x+0y &= 1 \end{aligned}$$

has matrix equation form

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

If A is the coefficient matrix of the system and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then we have that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that since $a_{2,1} = 0$ and that $i = 2$ and $j = 1$ are the only values of i and j in $\{1, 2\}$ for which $i > j$. Thus A is upper triangular. A also has only 1's and/or 0's on its main diagonal, since $a_{1,1} = 1$ and $a_{2,2} = 0$. Therefore, by the Upper Triangular Main Diagonal 1's and 0's Row Echelon Form Theorem, A is in row echelon form. Since $a_{1,1} = 1 \neq 0$, we have that Row 1 of A is non-zero. Since $a_{2,1} = a_{2,2} = 0$, Row 2 of A is not non-zero. Thus A has only one non-zero row and, since it is in row echelon form, then by definition of rank, $\text{rank}(A) = 1$.

Now let $D = A|\mathbf{b}$. Since $D \in M_{2\times 3}$, and $i = 2$ and $j = 1$ are the only values of $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ such that $i > j$, and since $d_{2,1} = 0$, we have that d is upper-triangular, by definition. Also, $d_{1,1} = 1$ and $d_{2,2} = 0$, so that it is also the case that D has only 1's and/or 0's on its main diagonal. Hence, by the Upper Triangular Main Diagonal 1's and 0's Row Echelon Form Theorem, D is in row-echelon form. Since $d_{1,1} = 1 \neq 0$ and $d_{2,3} = 1 \neq 0$, both rows of $D = A|\mathbf{b}$ are non-zero. Hence, by definition, we have that $\text{rank}(A|\mathbf{b}) = 2$. It follows that $\text{rank}(A) = 1 < 2 = \text{rank}(A|\mathbf{b})$. By Part(b) of the Rank Theorem for Complex Linear Systems of Equations, therefore, the system of equations giving rise to $A|\mathbf{b}$ must have no solution.

Exercise 163. Show that the system of equations in the above example has no solution without using the Rank Theorem for Complex Linear Systems of Equations.

Exercise 164. Use the Rank Theorem for Complex Linear Systems of Equations to show that

$$\begin{aligned} x + y + z &= 0 \\ x - y - z &= 1 \\ x - y - z &= -2 \end{aligned}$$

has no solutions in $\mathbb{R}^3$.

Exercise 165. Use the Rank Theorem for Complex Linear Systems of Equations to show that some non-real complex linear system of equations in at least two complex unknowns has exactly one solution.

Exercise 166. Solve the system of equations that you produced in the above exercise, this time using Gaussian elimination.

Example 19.8. The system of equations

$$\begin{aligned} x + 0y &= 0 \\ 0x + 0y &= 0 \end{aligned}$$

has matrix equation form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We showed in Example 19.7 that A has rank 1. The augmented matrix of the above system is

$$F = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Note that since $f_{1,1} = 1$ and $f_{2,2} = 0$, that F has only 1's and/or 0's on its main diagonal. Also, since $i = 2$ and $j = 1$ are the only values in $\{1, 2\}$ and $\{1, 2, 3\}$, respectively, such that $i > j$, and since $f_{2,1} = 0$, we have that F is upper triangular, by definition. It follows, therefore, by the Upper Triangular Main Diagonal 1's and 0's Row Echelon Form Theorem, that F is in row-echelon form. Since $f_{1,1} = 1 \neq 0$ and $f_{2,1} = f_{2,2} = f_{2,3} = 0$, we see that F has one row that is not non-zero. Thus $\text{rank}(A) = \text{rank}(F) = \text{rank}(A|\mathbf{b}) = 1 < 2$. Since A here is in $M_{2\times 2}(\mathbb{C})$, we have that n here is 2, and by part (c) of the Rank Theorem for Complex Linear Systems of Equations, the system of equations in this example has infinitely many solutions.

Exercise 167. Show that the system of linear equations in the above example has infinitely many solutions without using the Rank Theorem for Complex Linear Systems of Equations.

Exercise 168. Use the Rank Theorem for Complex Linear Systems of Equations to show that

$$\begin{aligned} x - 2y + 3z &= -4 \\ x + \ \ y + \ \ z &= 0 \\ -2x + 4y - 6z &= 8 \end{aligned}$$

has infinitely many solutions.

Exercise 169. Show that the above system of equations has infinitely many solutions without using the Rank Theorem for Complex Linear Systems of Equations. Find all such solutions.

Exercise 170. Prove the Upper Triangular Main Diagonal 1's and 0's Row-Echelon Form Theorem.

Definition 19.9. Suppose that $A \in M_{m\times n}(\mathbb{C})$ and $\overrightarrow{b} \in \mathbb{V}_m^{\mathbb{C}}$. Let $k \in \{1, 2, \ldots, n\}$. We say that x_k is a basic variable for the system of equations with augmented matrix $A|\mathbf{b}$ iff the system is consistent and column k of this augmented matrix is a pivot column. We call x_k a free variable iff it is not a basic variable.

Example 19.10. The system of linear equations in Example 19.8, as we saw, had augmented matrix F there that is already in row-echelon form. We showed that the system of equations was consistent, since it has infinitely many solutions. Since $f_{1,1} = 1$, then $f_{1,1} = 1$ is a leading 1 in F in the first column. Since the second column of F has nothing but 0's in it, F has no leading entry in its second column. Therefore, by definition, $x_1 = x$ is a basic variable for this system, and $x_2 = y$ is a free variable for this system.

Exercise 171. Determine the

(a) basic variable(s), and
(b) the free variable(s),

if there any, for the system of equations in Exercise 168. Of course you should prove your answer(s).

The following definition extends a similar idea that the reader has encountered in high school, in the case of real systems of linear equations.

Definition 19.11. A system of linear equations with complex coefficients is called *homogeneous* iff its augmented matrix is

$$A|\vec{0}_m \text{ for some } A \in M_{m\times n}(\mathbb{C})$$

Exercise 172. Find an example of a non-real complex system of linear equations in at least two variables, that is homogeneous.

Theorem 19.12 (Homogeneous Trivial Solution Theorem)**.** $\mathbf{0}_m$ is a solution of $A|\mathbf{0}_m$ $\forall m \in \mathbb{N}$, $n \in \mathbb{N}$, and $A \in M_{m\times n}(\mathbb{C})$.

Example 19.13. The Homogeneous Trivial Solution Theorem guarantees that $\mathbf{0}_2$ is a solution of the system of equations in Example 19.8.

Exercise 173. Show that $\mathbf{0}_2$ is a solution of the system of equations given in Example 19.8, without using the Homogeneous Trivial Solution Theorem.

Exercise 174. Find a homogeneous, non-real system of linear equations in at least 3 variables, and show that this system has at least one solution. What is that solution?

Exercise 175. Prove the Homogeneous Systems Trivial Solution Theorem.

The following definition should already be known to the reader in the case of real matrices:

Definition 19.14. The particular solution $\vec{x} = \vec{0}_n$ to a system $A\vec{x} = \vec{0}_m$ is called the <u>trivial solution</u> of the system.

Exercise 176. What is the trivial solution of

$$\begin{aligned} ix + y + z &= 0 \\ x - y - z &= 0 \\ x - y + z &= 0? \end{aligned}$$

Theorem 19.15 (Flat Homogeneous Augmented Matrix Infinite Solutions Theorem)**.** Suppose that $A|\vec{0}_m$ is the augmented matrix of a homogeneous system of linear equations with complex coefficients, where $A \in M_{m\times n}(\mathbb{C})$. If $m < n$, then the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

Example 19.16. Consider the system of equations

$$\begin{aligned} x+y+z &= 0 \\ x-y+z &= 0. \end{aligned}$$

The above system has matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the right side of the equation is $\mathbf{0}_2$, we have that our system of equations is homogeneous, with $m = 2$ here. Since the coefficient matrix here has $m = 2$ rows and $n = 3$ columns, with $2 < 3$, the system has infinitely many solutions, by the Flat Homogeneous Augmented Matrix Infinite Solutions Theorem.

Exercise 177. Use the Flat Augmented Matrix Homogeneous Infinite Solutions Theorem to show that

$$\begin{aligned} x-(2+i)y+1000z+3w &= 0 \\ 2x-3y+\pi z-10w &= 0 \end{aligned}$$

has infinitely many solutions. Find all of these solutions!

Exercise 178. Prove that the above system has infinitely many solutions without using the Flat Homogeneous Augmented Matrix Infinite Solutions Theorem.

Exercise 179. Prove the Flat Homogeneous Augmented Matrix Infinite Solutions Theorem.

20. Multiplicative Inverses of Complex Matrices

The following definitions should already be known to the reader in the case of real matrices. We are merely extending the definitions, with no changes other than allowing complex entries.

Definition 20.1. If $A \in M_{m\times n}(\mathbb{C})$, then $-A = (-a_{i,j})_{i,j=1}^{m,n}$ is called the <u>additive inverse of A</u>. If $m = n$, then we call B a <u>multiplicative inverse of A</u>, iff $AB = BA = I_n$.

Exercise 180. Find the additive inverse of

$$\begin{bmatrix} 1 & (1+2i) & -7 \\ -4-25i & 0 & -93. \end{bmatrix}$$

Exercise 181. Find a multiplicative inverse of

$$\begin{bmatrix} 1 & 2i \\ 3 & -4i \end{bmatrix}$$

if such a matrix exists. If such a matrix does not exist, prove that such a matrix does not exist. If such a matrix does exist, prove that the matrix obtained is a multiplicative inverse of the above matrix.

Theorem 20.2 (Matrix Multiplicative Inverse Uniqueness Theorem)**.** *Let $A \in M_n(\mathbb{C})$, $AB = I_n = BA$ and $AC = I_n = CA$ where $B, C \in M_n(\mathbb{C})$. Then $B = C$.*

Exercise 182. Show that the multiplicative inverse matrix found in the above exercise is the only multiplicative inverse of

$$\begin{bmatrix} 1 & 2i \\ 3 & -4i \end{bmatrix}.$$

Exercise 183. Suppose that $A, B \in M_n(\mathbb{C})$ and that $AB = I_n$. Is $B = A^{-1}$? Prove your answer.

Exercise 184. Suppose that $A, C \in M_n(\mathbb{C})$ and that $CA = I_n$. Is $C = A^{-1}$? Prove your answer.

Definition 20.3. We say that $A \in M_n(\mathbb{C})$ is (multiplicatively) invertible or *non-singular in* $M_n(\mathbb{C})$ iff $\exists B \in M_n(\mathbb{C})$ such that B is the multiplicative inverse of A. In this case, we denote B by A^{-1}. We otherwise call A *singular*.

Exercise 185. Denote the matrix in Exercise 181 by A. Show that A is non-singular and find A^{-1}.

Exercise 186. Prove that $0_{3\times 3}$ is singular in $M_3(\mathbb{C})$.

Exercise 187 (Research Project)**.** *To what extent can things in this chapter on linear algebra be transferred to the situation of other products of matrices, such as bracket, tensor, circle, direct, and the so-called tropical algebras? Linear algebra of tropical mathematics is a relatively unexplored area. For more information, google "Tropical Mathematics."*

Remark: It is important that reader note that $A^{-1} \neq \frac{1}{A}$ if A is a square complex matrix of order $n > 1$. However, if B is non-singular and $A, B \in M_n(\mathbb{C})$, then we can define

$$A/B := AB^{-1}$$

.

Exercise 188. Show that $1/A$ for a even a square complex matrix of order 2, makes no sense, according to anything that we have presented so far in these lecture notes.

The following theorem generalizes what the reader already knows, since precalculus algebra, in the case of real matrices, to the case of complex matrices.

Theorem 20.4 (Inverse Matrix Equation Existence and Uniqueness Theorem)**.** *If $[A|\vec{b}]$ is the augmented matrix of a system of linear equations with complex coefficients, where $A \in M_n(\mathbb{C})$ and A is non-singular, then there is one and only one solution to $A\vec{x} = \vec{b}$ and this solution $\vec{x}$ is given by*

$$\vec{x} = A^{-1}\vec{b}$$

Exercise 189. Use the Inverse Matrix Equation Existence and Uniqueness Theorem to find the solution of the following system:

$$\begin{aligned} x+y-z &= -1+2i \\ x-y-z &= -1 \\ x+2y+iz &= 4i. \end{aligned}$$

Theorem 20.5 (Rank-Invertibility Theorem)**.** If $A \in M_n(\mathbb{C})$, then A is non-singular iff $\text{rank}(A) = n$.

Example 20.6. Let A be the matrix in Exercise 181. We showed in that exercise that the matrix has a multiplicative inverse, so by definition, A is non-singular. $A \in M_2(\mathbb{C})$, so by the Rank-Invertibility Theorem, A has rank $n = 2$.

Exercise 190. Use the Rank-Invertibility Theorem to show that the coefficient matrix of the system in Exercise 189 has rank 3.

Theorem 20.7 (Matrix Equation Solvability/Invertibility Theorem)**.** Let $A \in M_n(\mathbb{C})$ and suppose that $\vec{b} \in \mathbb{V}_n^{\mathbb{C}}$. Assume that $A\vec{x} = \vec{b}$ has exactly one solution. Then A is non-singular.

Example 20.8. Since the system of linear equations in Exercise 161 has exactly one solution, then, by the Matrix Equation Solvability/Invertibility Theorem, we have that the coefficient matrix of that system is non-singular.

Exercise 191. Use the Matrix Equation Solvability/Invertibility Theorem to show that $\mathbf{0}_{2\times 2}\mathbf{x} = \mathbf{0}_2$ has infinitely many solutions.

The first two equations in the theorem that follows should already be known to the reader in the case of real matrices:

Theorem 20.9 (Non-singularity Properties)**.** *Assume that $A, B \in M_n(\mathbb{C})$ and that A, B are non-singular. Then the following statements hold:*

(1) $(A^{-1})^{-1} = A$.
(2) $(AB)^{-1} = B^{-1}A^{-1}$.
(3) $(A^{-1})^T = (A^T)^{-1}$.

Exercise 192. Find the multiplicative inverse of the multiplicative inverse of the multiplicative inverse of

$$\begin{bmatrix} i & 2 \\ 3i & 4+i \end{bmatrix}.$$

Exercise 193. Find

$$\left(\begin{bmatrix} i & 2 \\ 3i & 4+i \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right)^{-1}.$$

Example 20.10. The reader will recall that $I_2 I_2 = I_2$, so $I_2 = I_2^{-1}$, and we saw in an earlier section that $I_2^T = I_2$. By the third property of non-singular matrices, we have that $(I_2^T)^{-1} = (I_2^{-1})^T = I_2^T = I_2$.

Exercise 194. Find the transpose of the multiplicative inverse of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Corollary 20.11. *If $A_1, A_2, \ldots, A_j \in M_n(\mathbb{C})$ and are non-singular, then so is their matrix product. Furthermore, we have that*

$$(A_1 A_2 \ldots A_j)^{-1} = A_j^{-1} A_{j-1}^{-1} \ldots A_1^{-1}.$$

Theorem 20.12 (Non-singular Matrix Factorization Theorem)**.** Suppose that $A, B \in M_n(\mathbb{C})$. Assume that AB is a non-singular matrix. Then A and B are also non-singular.

Example 20.13. We have that

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} 1(1) + 2(3) & 1(2) + 2(4) \\ 3(1) + 4(3) & 3(2) + 4(4) \end{bmatrix} \\ &= \begin{bmatrix} 1 + 6 & 2 + 8 \\ 3 + 12 & 6 + 16 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}, \end{aligned}$$

which has determinant $7(22) - 10(15) = 154 - 150 = 4 \neq 0$. Recalling from precalculus that this non-zero determinant implies that the above matrix has a multiplicative inverse, we see that the matrix is non-singular. Hence, by the Non-Singular Matrix Factorization Theorem, it must be the case that both factors of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

must be non-singular. This implies that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

must be non-singular.

Exercise 195. Show that the above matrix is non-singular without using the determinant nor the above the theorem.

Exercise 196. Find two matrices A and $B \in M_3(\mathbb{C})$, both with all non-zero entries, such that $AB = I_3$. Use the above theorem to show that A and B are non-singular.

Please send any comments, questions, or corrections to me at:

E-mail address: `dclahane@fullcoll.edu`

Fullerton College, USA