Lecture 1 Problems

1*.

(a) *(Graphic matroids are linearly representable)*

Given a graph $G = (V, E)$, show that one can linearly represent its graphic matroid over any field $\mathbb{F}$ as follows. In the vector space $\mathbb{F}^V$ having standard basis vectors ϵ_i indexed by the vertices V in V, represent the element $e = \{v, v'\}$ in E by the vector $\epsilon_v - \epsilon_{v'}$.

In other words, show that the linearly independent subsets of these vectors are indexed by the forests of edges in G.

(b) *(Transversal linearly representable)*

Given a bipartite graph G with vertex bipartition $E \cup F$, show that one can represent its transversal matroid in any field characteristic as follows. Let $\mathbb{F}(x_{e,f})$ be a field extension of the field $\mathbb{F}$ by transcendentals $\{x_{e,f}\}$ indexed by all edges $\{e, f\}$ of G. Then in the vector space $\mathbb{F}(x_{e,f})^F$ having standard basis vectors ϵ_f indexed by the vertices f in F, represent the element $e \in E$ by the vector

$$\sum_{f \in F : \{e,f\} \in G} x_{e,f} \epsilon_f.$$

In other words, show that the linearly independent subsets of these vectors are indexed by the subsets of vertices in E that can be matched into F along edges of G.

(c) *(Linearly representable are algebraically representable)*

Given a matroid M of rank r linearly representd by a set of vectors $\{v_1, \ldots, v_n\}$ in the vector space $\mathbb{F}^r$, represent M algebraically by elements of the rational function field $\mathbb{F}(x_1, \ldots, x_r)$ as follows. If v_i has coordinates $(v_{i1}, \ldots, v_{ir})$ with respect to the standard basis for $\mathbb{F}^r$, then represent v_i by $f_i := \sum_{i=1}^{r} v_{ij} x_j$.

In other words, show that the algebraically independent subsets of these rational functions f_i are indexed the same as the linearly independent subsets of the v_i.

2*. *(Acyclic orientations and chambers)*

For a graph $G = (V, E)$, consider the hyperplane arrangement $\mathcal{A}$ having hyperplanes of the form $\{x_i = x_j\}_{\{i,j\}\in E}$. Explain how each top-dimensional cell (or *chamber* or *region*) in the decomposition of $\mathbb{R}^V$ cut out by the hyperplanes is naturally labelled by an acyclic orientation of the edges of G, and why this gives a bijection between the chambers and the acylic orientations.

3*. *(The greedy algorithm works for matroids)*

(a) Show that Kruskal's greedy algorithm (described in Lecture 1) always finds a maximum-weight independent set in a matroid, regardless of the choice of weight function $w : E \to \mathbb{R}_+$.

(b) Show that this property characterizes independent sets of matroids among all simplicial complexes. In other words, given a simplicial complex $\mathcal{I}$ for which the greedy algorithm always works, regarless of the weight function w, show that $\mathcal{I} = \mathcal{I}(M)$ for a matroid M.

(Hint: One only needs to show that the exchange axiom I3 holds. To do this, given I_1, I_2 in $\mathcal{I}$, with $|I_2| = |I_1| + 1 = k + 1$, consider the weight function

$$w(e) := \begin{cases} \frac{k+1}{k+2} & \text{for } e \in I_1 \\ \frac{k}{k+1} & \text{for } e \in I_2 - I_1 \end{cases} .$$

Explain why the greedy algorithm will build up I_1 first, and then at the next step, will exhibit an element of the form $I_1 \cup \{e\} \in \mathcal{I}$ with $e \in I_2 - I_1$. In particular, explain why the algorithm will not just stop after having found I_1!)

4. *(Circuit axioms are equivalent to independent set axioms)*

Recall that the circuit axioms assert that $\mathcal{C} \subseteq 2^E$ forms the *circuits* of a matroid M on the finite set E if

C1. $\varnothing \in \mathcal{C}$

C2. If $C, C' \in \mathcal{C}$ and $C \subset C'$, then $C = \mathcal{C}'$.

C3. Given $C, C' \in \mathcal{C}$ with $C \neq C'$ and $e \in C \cap C'$, there exists some $C'' \in \mathcal{C}$ with $C'' \subseteq C \cup C' - \{e\}$.

Show that circuits give an equivalent axiomatization of matroids as do independent sets.

In other words, given a collection $\mathcal{I} \subseteq 2^E$ of sets satisfying the independent set axioms, show that the collection $\mathcal{C}$ of minimal subsets not in $\mathcal{I}$ satisfy C1–C3, and conversely given a collection $\mathcal{C}$ satisfying C1–C3, show that the collection $\mathcal{I}$ of subsets containing no subset from $\mathcal{C}$ satisfy the independent set axioms.

5. *(Matroids are hereditarily pure)*

Show that restricting the independent sets of a matroid M on E to a subset E' of E always gives a pure simplicial complex on E'. Show that this property characterizes independent sets of matroids among all simplicial complexes.