Boundary Value Problems

• Scalar Expansions as Solutions of the Laplace's Equation:

- Laplace's Equation: $\nabla^2\Phi = 0$.
- Coefficients: A, B, C, D, E, F are coefficients, where α, β, m, l, k are the dummies.

- 2-D Box:

$$\Phi = \Sigma_\alpha^\infty (A_\alpha \cos\alpha x + B_\alpha \sin\alpha x)(C_\alpha \cosh\alpha y + D_\alpha \sinh\alpha y).$$

Boundary Conditions: $\Phi(x=0) = \Phi(x=a) = 0$ gives,

$$\Phi = \Sigma_{n=0}^\infty \sin\frac{n\pi}{a}x\left(C_n\cosh\frac{n\pi}{a}y + D_n\sinh\frac{n\pi}{a}y\right).$$

- 2-D Polar:

$$\Phi = (A_0 + B_0\ln\rho)(C_0 + D_0\phi) + \Sigma_\alpha^\infty (A_\alpha\rho^\alpha + B_\alpha\rho^{-\alpha})(C_\alpha\cos\alpha\phi + D_\alpha\sin\alpha\phi).$$

Boundary Conditions: $\Phi(\phi=0) = \Phi(\phi=2\pi)$ gives,

$$\Phi = A_0 + B_0\ln\rho + \Sigma_{m=1}^\infty (A_m\rho^m + B_m\rho^{-m})C_m\cos m\phi.$$

- 3-D Box:

$$\Phi = \Sigma_\alpha^\infty\Sigma_\beta^\infty (A_{\alpha\beta}\cos\alpha x + B_{\alpha\beta}\sin\alpha x)(C_{\alpha\beta}\cos\beta y + D_{\alpha\beta}\sin\beta y)(E_{\alpha\beta}\cosh\sqrt{\alpha^2+\beta^2}z + F_{\alpha\beta}\sinh\sqrt{\alpha^2+\beta^2}z).$$

Boundary Conditions: $\Phi(x=0) = \Phi(x=a) = \Phi(y=0) = \Phi(y=b) = 0$ gives,

$$\Phi = \Sigma_{n=0}^\infty\Sigma_{m=0}^\infty \sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\left(E_{nm}\cosh\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}z + F_{nm}\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}z\right).$$

- Spherical Harmonic:

$$\Phi = \Sigma_{l=0}^\infty\Sigma_{m=-l}^l (A_{lm}r^l + B_{lm}r^{-l-1})Y_{lm}(\theta,\phi).$$

Boundary Conditions: azimuthal symmetry, i.e. $\Phi(\phi) = \Phi(0)\ \forall\phi$, gives, Legendre Function

$$\Phi = \Sigma_{l=0}^\infty (A_l r^l + B_l r^{-l-1})P_l(\cos\theta).$$

- Ordinary Bessel Function:

$$\Phi = \Sigma_{m=0}^\infty \int_0^\infty dk[A_m(k)J_m(k\rho) + B_m(k)N_m(k\rho)][C_m(k)\cos m\phi + D_m(k)\sin m\phi][E_m(k)e^{kz} + F_m(k)e^{-kz}].$$

Boundary Conditions: $\Phi(\rho=0)$ and $\Phi(z\to\infty)$ finite, and $\Phi(\rho=a)=0$ gives,

$$\Phi = \Sigma_{m=0}^\infty\Sigma_{n=1}^\infty J_m(\frac{x_{mn}}{a}\rho)[C_{mn}\cos m\phi + D_{mn}\sin m\phi]e^{-\frac{x_{mn}}{a}z},$$

where $J_m(x_{mn}) = 0$.

- Modified Bessel Function:

$$\Phi = \Sigma_{m=0}^\infty \int_0^\infty dk[A_m(k)I_m(k\rho) + B_m(k)K_m(k\rho)][C_m(k)\cos m\phi + D_m(k)\sin m\phi][E_m(k)\cos kz + F_m(k)\sin kz].$$

Boundary Conditions: azimuthal symmetry, $\Phi(\rho\to\infty)$ finite, and$\Phi(z=0) = \Phi(z=L) = 0$ gives,

$$\Phi = \Sigma_{m=0}^\infty\Sigma_{n=1}^\infty A_{mn}K_m\left(\frac{n\pi}{L}\rho\right)\cos m\phi\sin\left(\frac{n\pi}{L}z\right).$$

- **Orthogonality:**

$$\int_0^a \sin\frac{n\pi}{a}x\sin\frac{n'\pi}{a}xdx = \int_0^a \cos\frac{n\pi}{a}x\cos\frac{n'\pi}{a}xdx = \frac{a}{2}\delta_{n,n'}$$

$$\int_0^{2\pi}\int_\pi^0 Y^*_{l,m}(\theta,\phi)Y_{l,m}(\theta,\phi)d(\cos\theta)d\phi = \delta_{l,l'}\delta_{m,m'}$$

$$\int_1^{-1} P_l(\cos\theta)P_{l'}(\cos\theta)d\cos\theta = \frac{2}{2l+1}\delta_{l,l'}$$

$$\int_0^\infty J_m(k\rho)J_m(k\rho)\rho d\rho = \frac{1}{k}\delta(k-k')$$

$$\int_0^\alpha J_m\left(\frac{x_{mn}}{a}\rho\right)J_m\left(\frac{x_{mn'}}{a}\rho\right)\rho d\rho = \frac{a^2}{2}[J_{m+1}(x_{mn})]^2\delta_{n,n'}$$

- **Scalar 1/r Expansions:**

- 2-D Box:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\pi}\int d^2k\frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2} = \frac{1}{\pi}\int d\alpha d\beta\frac{e^{i\alpha(x-x')+i\beta(y-y')}}{\alpha^2+\beta^2} = \int d\alpha\frac{e^{i\alpha(x-x')}e^{-\alpha(y-y')}}{\alpha}$$

- 2-D Polar:

$$\ln\frac{1}{|\vec{x}-\vec{x}'|} = \ln\frac{1}{\rho_<} + \Sigma_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos m(\phi-\phi')$$

- 3-D Box:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{2\pi^2}\int d^3k\frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2} = \frac{1}{2\pi}\int d\alpha d\beta\frac{e^{i\alpha(x-x')}e^{i\beta(y-y')}e^{-\sqrt{\alpha^2+\beta^2}(z-z')}}{\alpha^2+\beta^2}$$

- Spherical Harmonic:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \Sigma_{l=0}^{\infty}\Sigma_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}\right)Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

- Legendre Function:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \Sigma_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}}\right)P_l(\cos\theta')P_l(\cos\theta)$$

- Ordinary Bessel Function:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \int_0^\infty dk\left(\frac{1}{2}J_0(k\rho)J_0(k\rho') + \Sigma_{m=1}^{\infty}J_m(k\rho)J_m(k\rho')\cos m(\phi-\phi')\right)e^{-k(z_>-z_<)}$$
$$= \Sigma_{m=-\infty}^{\infty}\int_0^\infty dk\left(J_m(k\rho)J_m(k\rho')e^{im(\phi-\phi')}\right)e^{-k(z_>-z_<)}$$

- Modified Bessel Function:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{4}{\pi}\int_0^\infty dk\left(\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \Sigma_{m=1}^{\infty}I_m(k\rho_<)K_m(k\rho_>)\cos m(\phi-\phi')\right)\cos k(z-z')$$
$$= \frac{4}{\pi}\Sigma_{m=-\infty}^{\infty}\int_0^\infty dk\left(I_m(k\rho_<)K_m(k\rho_>)e^{im(\phi-\phi')}\right)\cos k(z-z')$$

Green's Functions

• **Basics:**

- Electric Potential in Green's Function (Jackson Eq. 1.42):

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')d^3x' + \frac{1}{4\pi}\oint_{\partial V}\frac{\partial\Phi(\vec{x}')}{\partial n'}G(\vec{x},\vec{x}') - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}d^2a'$$

- Reciprocity Theorem:

$$\int_V \rho(\vec{x})\Phi'(\vec{x}')d^3x + \int_{\partial V}\sigma(\vec{x})\Phi'(\vec{x}')d^2a = \int_V \rho'(\vec{x})\Phi(\vec{x}')d^3x + \int_{\partial V}\sigma'(\vec{x})\Phi(\vec{x}')d^2a$$

$$\int_V \rho\Phi' + \int_{\partial V}\sigma\Phi' = \int_V \rho'\Phi + \int_{\partial V}\sigma'\Phi$$

- Method to Find Green's Functions: propose a geometric shape. Make the shells all grounded conductors and the spaces all vacuum. Impose a test charge in the region of question. Divid the regions by the shells according to the coordinate chosen, and also divid the region across the test charge. Then solve for the potential in the region of question (in the $>$ and $<$ manner, where the $\vec{x}'$ goes for the source, i.e. the test charge) as boundary value problem. Note the the only discontinuity for the boundary conditions is on the boundary where the test charge is located. Finally, plug in $q = 4\pi\varepsilon_0$.
- IMPORTANT Method: imposing the source term with $\frac{1}{|\vec{x}-\vec{x}'|}$ expansion and then matching the terms of the expansion (matches the direction as well for the vector potentials) for the boundary value problem, is often easier than making another boundary across the source terms.
- Non-orthogonality Conditions: should be applied to the boundary that decides "$>$" and "$<$", e.g. sinh and cosh are not orthogonal and so are used when cylinder is divided in z-direction, as in ordinary Bessel, while modified Bessel divids the cylinder in ρ-direction.

• **Some Green's Functions:** "$>$" and "$<$" are comparison of $\vec{x}$ and $\vec{x}'$ and they symbolizes the "jumps" in Green's functions,

- Inside the x-y Direction of a Box:

$$G(\vec{x},\vec{x}') = \Sigma_{n=1}^{\infty}\Sigma_{m=1}^{\infty}\frac{16\pi}{ab}\sin\frac{n\pi}{a}x\sin\frac{n\pi}{a}x'\sin\frac{m\pi}{b}y\sin\frac{m\pi}{b}y'\frac{\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}z_<\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}(c-z_>)}{\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}c}.$$

- The Intermediate Region of Concentric Sphere (a $<$ b):

$$G(\vec{x},\vec{x}') = \Sigma_{l=0}^{\infty}\Sigma_{m=-l}^{l}\frac{4\pi}{2l+1}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)\frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{1-(\frac{a}{b})^{2l+1}}.$$

- Inside the Radius of a Finite Cylinder with Radius "a" and Height "L" (Ordinary Bessel):

$$G(\vec{x},\vec{x}') = \Sigma_{m=0}^{\infty}\Sigma_{n=1}^{\infty}\frac{8}{ax_{mn}}\frac{J_m(\frac{x_{mn}}{a}\rho)J_m(\frac{x_{mn}}{a}\rho')}{(J_{m+1}(x_{mn}))^2}\frac{2\cos m(\phi-\phi')}{1+\delta_{m,0}}\frac{\sinh\left(\frac{x_{mn}}{a}(L-z_>)\right)\sinh\left(\frac{x_{mn}}{a}z_<\right)}{\sinh\left(\frac{x_{mn}}{a}L\right)}.$$

- Inside the Height of a Finite Cylinder with Radius "a" and Height "L" (Modified Bessel):

$$G(\vec{x},\vec{x}') = \Sigma_{m=0}^{\infty}\Sigma_{n=1}^{\infty}\frac{8}{L}\frac{I_m(\frac{n\pi}{L}\rho)\left[K_m\left(\frac{n\pi}{L}a\right)I_m\left(\frac{n\pi}{L}\rho_>\right) - K_m\left(\frac{n\pi}{L}\rho_<\right)I_m\left(\frac{n\pi}{L}a\right)\right]}{I_m(\frac{n\pi}{L}a)}\frac{2\cos m(\phi-\phi')}{1+\delta_{m,0}}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right).$$

Paralleling Electrostatics and Magnetostatics

	Electrostatics	Magnetostatics
	$\vec{E} = -\vec{\nabla}\Phi$	$\vec{B} = \vec{\nabla} \times \vec{A}$
		$\vec{H} = -\vec{\nabla}\Phi_M$
	$\Phi = \frac{1}{4\pi\varepsilon} \int \frac{\rho(\vec{x}')}{\lvert\vec{x} - \vec{x}'\rvert} d^3x'$	$\vec{A} = \frac{\mu}{4\pi} \int \frac{\vec{J}(\vec{x}')}{\lvert\vec{x} - \vec{x}'\rvert} d^3x'$
	$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$	$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$
	$\vec{\nabla} \cdot \vec{D} = \rho$	$\vec{\nabla} \times \vec{H} = \vec{J}$
		$\vec{\nabla} \times \vec{M} = \vec{J_m}$, $\vec{M} \times \hat{n} = \vec{K}_m$
	$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = \sigma$, $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$	$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0$, $\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}$
linear di-ic	$\vec{D} = \varepsilon\vec{E} = (1 + \chi_e)\varepsilon_0\vec{E}$	$\vec{H} = \frac{1}{\mu}\vec{B} = (1 + \chi_m)\frac{1}{\mu_0}\vec{B}$
dipole	$\vec{p} = \int \vec{x}' \rho(\vec{x}') d^3x'$	$\vec{m} = \frac{1}{2} \int \vec{x}' \times \vec{J}(\vec{x}') d^3x'$
	$\vec{p} = \int \vec{P}(\vec{x'}) d^3x'$	$\vec{m} = \int \vec{M}(\vec{x'}) d^3x'$
far	$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon} \frac{3\hat{n}(\vec{p} \cdot \hat{n} - \vec{p})}{\lvert\vec{x} - \vec{x}_0\rvert^3}$	$\vec{B}(\vec{x}) = \frac{\mu}{4\pi} \frac{3\hat{n}(\vec{m} \cdot \hat{n} - \vec{m})}{\lvert\vec{x} - \vec{x}_0\rvert^3}$
	$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon} \frac{(\vec{p} \cdot \vec{x})}{\lvert\vec{x} - \vec{x}_0\rvert^3}$	$\vec{A}(\vec{x}) = \frac{\mu}{4\pi} \frac{(\vec{m} \times \vec{x})}{\lvert\vec{x} - \vec{x}_0\rvert^3}$
force	$\vec{F} = \int \rho(\vec{x})\vec{E}(\vec{x}) d^3x$	$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x}) d^3x$
		$\vec{\tau} = \int \vec{x} \times \big(\vec{J}(\vec{x}) \times \vec{B}(\vec{x})\big) d^3x$
	$\vec{F} = \vec{\nabla}(\vec{p} \cdot \vec{E})$	$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$
		$\vec{\tau} = \vec{m} \times \vec{B}$
work	$W = \frac{1}{2} \int \rho(\vec{x})\Phi(\vec{x}) d^3x$	$W = \frac{1}{2} \int \vec{J}(\vec{x}) \cdot \vec{A}(\vec{x}) d^3x$
	$W = \frac{1}{2}\Sigma_i\Sigma_j P_{ij} Q_i Q_j$	$W = \frac{1}{2}\Sigma_i L_i I_i^2 + \Sigma_i\Sigma_j M_{ij} I_i I_j$
	$W = \frac{1}{2}\Sigma_i\Sigma_j C_{ij} V_i V_j$	
linear di-ic	$W = \frac{1}{2} \int \vec{E}(\vec{x}) \cdot \vec{D}(\vec{x}) d^3x$	$W = \frac{1}{2} \int \vec{H}(\vec{x}) \cdot \vec{B}(\vec{x}) d^3x$

Others:
- Current Density and Current: $\vec{J} = \rho\vec{v}$, $I = \oint \vec{J} \cdot d\vec{a}$, where $\vec{a}$ is the cross section.
- Scalar Potential Under Permanent Magnetization: $\Phi_M(\vec{x}) = -\frac{1}{4\pi} \int_V \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{4\pi} \oint_{\partial V} \frac{\hat{n} \cdot \vec{M}(\vec{x})}{|\vec{x} - \vec{x}'|} d^2a'$.
- For source terms, integrate over uniform distribution with delta functions to get the pre-factors.

Electrodynamics: General Radiation

- **Maxwell's Equations:**

$$\text{Coulomb:} \quad \vec{\nabla} \cdot \vec{D} = \rho \quad \text{Faraday:} \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\text{NoMagMono:} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \text{Ampere:} \quad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}.$$

- **Continuity Equation:**

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0.$$

- **Potentials:**

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi.$$

- **Relations:**

- Propagation regions are usually sourceless. And without the closed boundary condition in any certain direction (i.e. wave guides), the only spacial dependent of the field is the variable in the propagation direction.
- NoMagMono/Faraday are homogeneous (source-unrelated), and are associated with gauge invariance.
- Coulomb/Ampere are inhomogeneous (source-related), and are associated with continuity equation. Get the relation between $\vec{A}$ and Φ by plugging Coulomb/Ampere into the continuity equation.
- Wave equations of the fields are obtain by taking curl of Faraday/Ampere, and use $\vec{\nabla} \times \vec{\nabla} \times \vec{a} = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \vec{\nabla}^2 \vec{a}$. Notice that n is a relation between k and ω, which is derived through fields' wave equations.
- Wave equations of the potentials are obtained by taking divergence of Coulomb and taking curl of Ampere.
- Relation between $\vec{E}$ and $\vec{B}$ are obtained by Faraday/Ampere. The curl should be taken on which ever is given. This relation includes the perpendicularity of $\vec{E}$ and $\vec{B}$.
- To test the fields are transverse, first set up the propagation direction (in the exponential), then use Coulomb/NoMagMono, gauge condition after plugging in the potentials, or the relation between $\vec{E}$ and $\vec{B}$ to show that $\hat{k} \cdot \vec{E} = 0$ or $\hat{k} \cdot \vec{B} = 0$. For propagating wave in free space, $\vec{B}$ and $\vec{D}$ are always transverse.
- The field are usually in the transverse direction, and the vector potential is in the same direction as the electric field.

- **Gauge Invariance:**

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \qquad \Phi \to \Phi - \frac{\partial \vec{\Lambda}}{\partial t}.$$

- **Gauges:**

$$\text{Coulomb:} \quad \vec{\nabla} \cdot \vec{A} = 0 \quad \text{Lorentz:} \quad \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0.$$

- **Retarded Green's Function:**

$$G^{(+)}(\vec{x}, t; \vec{x'}, t') = \frac{\delta\big(t' - [t - \frac{|\vec{x}-\vec{x}'|}{c}]\big)}{|\vec{x} - \vec{x}'|}.$$

$$\psi(\vec{x}, t) = \int \frac{[f(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|} d^3x',$$

where $(\psi, f) = (\vec{A}, \vec{J})$, (Φ, ρ), and $[f(\vec{x}', t')]_{\text{ret}} = \int \delta\big(t' - [t - \frac{|\vec{x}-\vec{x}'|}{c}]\big) f(\vec{x}', t') dt'$.
- Useful delta relation:

$$\delta(F(\vec{x})) = \sum \frac{\delta(\vec{x} - \vec{x}_i)}{|\frac{\partial F}{\partial x}|_{\vec{x}=\vec{x}_i}}, \; F(\vec{x}_i) = 0.$$

- **Jefimenko:** $\vec{R} = |\vec{x} - \vec{x}'|$

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \left(\frac{\hat{R}}{R^2}[\rho(\vec{x}', t')]_{\text{ret}} + \frac{\hat{R}}{cR}\left[\frac{\partial \rho(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} - \frac{1}{c^2 R}\left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} \right)$$

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \left([\vec{J}(\vec{x}', t')]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} \times \frac{\hat{R}}{cR} \right).$$

- **Poynting:**

$$W_{\text{by field}} = \int \vec{J} \cdot \vec{E} \quad \Rightarrow \frac{du}{dt} + \vec{\nabla} \cdot \vec{S} + \vec{J} \cdot \vec{E} = 0, \qquad u \equiv \frac{1}{2}(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \quad \text{(energy)}$$

$$\vec{S} \equiv \vec{E} \times \vec{H} \quad \text{(energy flow)}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \Rightarrow \vec{P}_{\text{field}} = \int \vec{g}, \qquad \vec{g} = \mu\varepsilon \vec{E} \times \vec{H} = \frac{\vec{S}}{c} \quad \text{(momentum)}$$

$$\Rightarrow \frac{d}{dt}(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}})_\alpha = \oint T_{\alpha\beta}\eta_\beta, \quad T_{\alpha\beta} = \varepsilon(E_\alpha E_\beta + \frac{B_\alpha B_\beta}{\mu\varepsilon} - \frac{E^2 + \frac{B^2}{\mu\varepsilon}}{2}\delta_{\alpha\beta}) \quad \text{(stress)}.$$

- **Invariance:**

$\vec{E}, \vec{P}, \vec{D}, \vec{\nabla}$	rank 1, odd vec	even time
$\vec{B}, \vec{M}, \vec{H}$	rank 1, even vec	odd time
ρ	rank 0, even scal	even time
$\vec{J}$	rank 1, odd vec	odd time.

- Cross product does not vary the ranks, but dot product does.
- If rank, trans, time symmetry match, then such term can potentially be in the Taylor expansion.
- Continuously multiplying the scaler terms can always be absorbed into the coefficient of the Taylor expansion.

- **Sourceless Maxwell's Equations:** describes how light passes through different media,

$$\text{Coulomb:} \quad \vec{\nabla} \cdot \vec{D} = 0 \quad \text{Faraday:} \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\text{NoMagMono:} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \text{Ampere:} \quad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0.$$

If $\vec{E} = \vec{E}_0 e^{i\vec{k}\cdot x - i\omega t}$, then $\vec{B}_0 = \sqrt{\mu\varepsilon}\hat{k} \times \vec{E}_0$.

$$\omega = kv,\ v = \frac{1}{\sqrt{\mu\varepsilon}},\ n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}};\ \omega = \frac{2\pi}{T} = 2\pi f,\ k = \frac{2\pi}{\lambda}.$$

- **Time Average of the Real Part:**

$$\frac{\int_0^{\frac{2\pi}{\omega}} \cos^2 \omega t dt}{\int_0^{\frac{2\pi}{\omega}} dt} = \frac{1}{2}.$$

$$\vec{S} \cdot \hat{n} = \frac{1}{2}\text{Re}[\hat{n} \cdot \vec{E} \times \vec{H}^*].$$

- **Polarization:** relation between the transverse fields, suppose propagation in z direction,

- Transverse (linear):

$$\vec{\epsilon}_1 = \hat{x},\ \vec{\epsilon}_2 = \hat{y}.$$

- Circular:

$$\vec{\epsilon}_\pm = \vec{\epsilon}_1 \pm \vec{\epsilon}_2.$$

- Stoke: only need 3 s's to completely specify the polarization.

from linear	from circular
$s_0 = \lvert\vec{\epsilon}_1 \cdot \vec{E}\rvert^2 + \lvert\vec{\epsilon}_2 \cdot \vec{E}\rvert^2$	$s_0 = \lvert\vec{\epsilon}_+^* \cdot \vec{E}\rvert^2 + \lvert\vec{\epsilon}_-^* \cdot \vec{E}\rvert^2$
$s_1 = \lvert\vec{\epsilon}_1 \cdot \vec{E}\rvert^2 - \lvert\vec{\epsilon}_2 \cdot \vec{E}\rvert^2$	$s_1 = 2\text{Re}[(\vec{\epsilon}_+^* \cdot \vec{E})^*(\vec{\epsilon}_-^* \cdot \vec{E})]$
$s_2 = 2\text{Re}[(\vec{\epsilon}_1 \cdot \vec{E})^*(\vec{\epsilon}_2 \cdot \vec{E})]$	$s_2 = 2\text{Im}[(\vec{\epsilon}_+^* \cdot \vec{E})^*(\vec{\epsilon}_-^* \cdot \vec{E})]$
$s_3 = 2\text{Im}[(\vec{\epsilon}_1 \cdot \vec{E})^*(\vec{\epsilon}_2 \cdot \vec{E})]$	$s_3 = \lvert\vec{\epsilon}_+^* \cdot \vec{E}\rvert^2 - \lvert\vec{\epsilon}_-^* \cdot \vec{E}\rvert^2.$

- **Boundary Conditions:** E for transmitted, E' for transmitted, E" for reflected,

$$(\vec{D} + \vec{D}'' - \vec{D}') \cdot \hat{n} = 0 \quad \text{(linear di-ic)} \Rightarrow \quad (\varepsilon\vec{E} + \varepsilon\vec{E}'' - \varepsilon'\vec{E}') \cdot \hat{n} = 0$$

$$(\vec{B} + \vec{B}'' - \vec{B}') \cdot \hat{n} = 0 \quad (\vec{k} \times \vec{D} + \vec{k} \times \vec{E}'' - \vec{k}' \times \vec{E}') \cdot \hat{n} = 0$$

$$(\vec{E} + \vec{E}'' - \vec{E}') \times \hat{n} = 0 \quad (\vec{E} + \vec{E}'' - \vec{E}') \times \hat{n} = 0$$

$$(\vec{H} + \vec{H}'' - \vec{H}') \times \hat{n} = 0 \quad (\frac{1}{\mu}\vec{k} \times \vec{D} + \frac{1}{\mu}\vec{k} \times \vec{E}'' - \frac{1}{\mu'}\vec{k} \times' \vec{E}') \times \hat{n} = 0.$$

- **Snell:** i is the incident angle, and r is the refracted angle.

$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \frac{n'}{n},\ k'' = -k.$$

- **Rate:** T + R = 1.

- Transmission Rate: $T \equiv \frac{|\vec{S}'|}{|\vec{S}|} = \frac{\frac{1}{2}|\vec{E}'\times\vec{H}'^*|}{\frac{1}{2}|\vec{E}\times\vec{H}^*|} = \frac{\sqrt{\mu'\varepsilon'}}{\mu'}\frac{1}{\frac{\sqrt{\mu\varepsilon}}{\mu}}\left|\frac{\vec{E}'}{\vec{E}}\right|$.

- Reflection Rate: $R \equiv \frac{|\vec{S}''|}{|\vec{S}|} = \left|\frac{\vec{E}''}{\vec{E}}\right|$.

- Fresnel Equations: i is the incident angle, and r is the refracted angle.

$\vec{E} \perp$ plane of incidence:

$$E_0 + E_0'' - E_0' = 0$$

$$\sqrt{\frac{\varepsilon}{\mu}}E_0\cos i - \sqrt{\frac{\varepsilon}{\mu}}E_0''\cos i - \sqrt{\frac{\varepsilon'}{\mu'}}E_0'\cos r = 0$$

$$\Rightarrow \frac{E_0'}{E_0} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin i}}$$

$$\frac{E_0''}{E_0} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin i}}$$

$\vec{E} \parallel$ plane of incidence:

$$E_0\cos i - E_0''\cos i - E_0'\cos r = 0$$

$$\sqrt{\frac{\varepsilon}{\mu}}E_0 + \sqrt{\frac{\varepsilon}{\mu}}E_0'' - \sqrt{\frac{\varepsilon'}{\mu'}}E_0' = 0$$

$$\frac{E_0'}{E_0} = \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin i}}$$

$$\frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin i}}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin i}}$$

- **Brewster's Angle:** R = 0.

$$\varepsilon = \varepsilon', \quad \text{only possible in } E \perp \text{ plane of incidence}$$
$$\mu = \mu', \quad \text{only possible in } E \parallel \text{ plane of incidence.}$$

- **Total Internal Reflection:**

$$n > n',\ i > i_0 = \sin^{-1}(\frac{n'}{n}).$$

- **Frequency Dispersion (Simple Model):** ω is the driven frequency, μ is not ω dependent.

$$\varepsilon(\omega) = \varepsilon_0 + \frac{Ne^2}{m_e}\sum_j^Z(\omega_j^2 - \omega^2 - i\omega\gamma_j)^{-1},$$

where N is # of molecules and Z is # of electron in one molecule. The γ_j are damping constants. In a wave function, attenuation constant α indicates an exponential decay:

$$k = \beta + i\frac{\alpha}{2}, \text{ where } \beta^2 - \frac{\alpha^2}{4} = \frac{\omega^2}{c^2}\text{Re}[\frac{\varepsilon}{\varepsilon_0}],\ \beta\alpha = \frac{\omega^2}{c^2}\text{Im}[\frac{\varepsilon}{\varepsilon_0}].$$

- Conductivity σ: $\varepsilon_{\text{total}}(\omega) = \varepsilon_{\text{(bounded)}}(\omega) + i\frac{\sigma}{\omega}$. Ohm's law: $\vec{J}(\vec{x},\omega) = \sigma(\omega)\vec{E}(\vec{x},\omega)$.
Poor conductor: $\frac{\sigma}{2\varepsilon} \ll \frac{k}{\sqrt{\mu\varepsilon}}$; Good conductor: $\frac{\sigma}{2\varepsilon} \gg \frac{k}{\sqrt{\mu\varepsilon}}$, $\gamma_{\text{(free)}} \approx 0$ (Start from $\frac{k^2}{\omega^2} = \frac{1}{\mu\varepsilon}$).

- Low frequency limit ($\omega \ll \max(\omega_j)$): $\varepsilon(\omega) \approx \varepsilon_{\text{(bounded)}} + i\frac{NZ_{\text{(free)}}e^2}{m_e\omega\left(\gamma_{\text{(free)}} - i\omega\right)}$.

- High frequency limit ($\omega \gg \max(\omega_j)$): $\varepsilon(\omega) \approx \varepsilon_0 - \varepsilon_0\frac{\omega_p^2}{\omega^2}$, where $\omega_p^2 = \frac{NZe^2}{\varepsilon_0 m_e}$; $k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$.

- Kramers-Kronig Relations: linear response of $\vec{D}(\vec{x},t)$ under causality of $\varepsilon(\omega)$, $\vec{D}(\vec{x},\omega) = \varepsilon(\omega)\vec{E}(\vec{x},\omega)$.

- Phase velocity $\vec{v}_p = \frac{\omega}{k}\hat{k}$; group velocity $\vec{v}_g = \frac{d\omega}{dk}\hat{k}$.

Electrodynamics: Waveguides & Resonance Cavities

• **Basics:** S indicates boundary, $\hat{n}$ is the unit normal of the boundaries; ω is the driven frequency (externally imposed), k_z with $\pm\hat{z}$ being the propagating direction. Also the following of fields are assumed (t for transverse):

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}\vec{E}(x,y)\\ \vec{B}(x,y)\end{Bmatrix} e^{\pm ik_z z - i\omega t} = (\nabla_t^2 - k_z^2 + \mu\varepsilon\omega^2)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0.$$

	TM mode ($B_z = 0$)	TE mode ($E_z = 0$)
Useful BCs:	$E_z\vert_S = 0$	$\left.\frac{\partial B_z}{\partial n}\right\vert_S = 0$
Impedance:	$Z = \frac{k_z}{\varepsilon\omega}$	$Z = \frac{\mu\omega}{k_z}$
Works for all following cases:	$\vec{E}_t = \frac{\pm ik_z}{\mu\varepsilon\omega^2 - k_z^2}\vec{\nabla}_t E_z$	$\vec{H}_t = \frac{\pm ik_z}{\mu\varepsilon\omega^2 - k_z^2}\vec{\nabla}_t H_z$
True only for $e^{\pm ik_z z - i\omega t}$:	$\vec{H}_t = \frac{\pm 1}{Z}\hat{z}\times\vec{E}_t$	$\vec{E}_t = \mp Z\hat{z}\times\vec{H}_t.$

• **Non-perfect Conductor and (Time Averaged) Power Loss:** σ is conductivity and c stands for conductor. With skin depth $\delta = \sqrt{\frac{2}{\mu_c\omega\sigma}}$, ξ indicates the depth into the conductor.

$$\vec{H}_c = \vec{H}_{\parallel} e^{-\frac{\xi}{\delta}} e^{i\frac{\xi}{\delta}},\ \vec{E}_c \approx \frac{1}{\delta\sigma}(1-i)\hat{n}\times\vec{H}_{\parallel} e^{-\frac{\xi}{\delta}} e^{i\frac{\xi}{\delta}}.$$

- Waveguides: β_λ is the attenuation constant and $\omega_\lambda \equiv \sqrt{\omega^2 - \frac{k^2}{\mu\varepsilon}}$. The power lost when the fields propagates is,

$$P(z) = P_0 e^{-2\beta_\lambda z},\ \beta_\lambda = -\frac{1}{2P}\frac{dP}{dz},\ \frac{dP_{\text{loss}}}{dz} = -\frac{dP}{dz},\ v_g = \frac{P}{U}.$$

	TM mode	TE mode
Energy	$U = \frac{1}{2}\frac{\omega^2}{\omega_\lambda^2}\varepsilon\int_S \vert E_z\vert^2 da$	$U = \frac{1}{2}\frac{\omega^2}{\omega_\lambda^2}\mu\int_S \vert H_z\vert^2 da$
Power	$P = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\frac{\omega^2}{\omega_\lambda^2}\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}\int_S \vert E_z\vert^2 da$	$P = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\frac{\omega^2}{\omega_\lambda^2}\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}\int_S \vert H_z\vert^2 da$
	$\frac{dP_{\text{loss}}}{dz} = \frac{1}{2\delta\sigma}\frac{\omega^2}{\omega_\lambda^2}\oint_{\partial S}\frac{1}{\mu^2\omega_\lambda^2}\left\vert\frac{\partial E_z}{\partial n}\right\vert^2 dl$	$\frac{dP_{\text{loss}}}{dz} = \frac{1}{2\delta\sigma}\frac{\omega^2}{\omega_\lambda^2}\oint_{\partial S}\frac{1}{\mu\varepsilon\omega_\lambda^2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)\vert\hat{n}\times\vec{\nabla}_t H_z\vert^2 + \frac{\omega_\lambda^2}{\omega^2}\vert H_z\vert dl.$

- Resonance Cavity: $Q \equiv \omega_0\frac{\text{Stored Energy}}{\text{Power Loss}}$ and $\gamma_\lambda \equiv \sqrt{\mu\varepsilon\omega^2 - \left(\frac{p\pi}{L}\right)^2}$. The energy lost over time is,

$$U(t) = U_0 e^{-\omega_0\frac{t}{Q}},\ \frac{1}{Q} = -\frac{1}{\omega_0 U}\frac{dU}{dt},\ P_{\text{loss}} = -\frac{dU}{dt}.$$

	TM mode	TE mode
Energy	$U = \frac{L}{4}\left[1+\left(\frac{p\pi}{\gamma_\lambda L}\right)^2\right]\varepsilon\int_S \vert E_z\vert^2 da$	$U = \frac{L}{4}\left[1+\left(\frac{p\pi}{\gamma_\lambda L}\right)^2\right]\mu\int_S \vert H_z\vert^2 da$

$$P_{\text{loss}} = \frac{1}{2\delta\sigma}\left[\oint_{\partial S}\int_0^L \vert\hat{n}\times\vec{H}\vert^2_{\text{sides}} dz\,dl + 2\int_S \vert\hat{n}\times\vec{H}\vert^2_{\text{ends}} da\right].$$

For the following, $e^{-i\omega t}$ will be neglected for convenience. Notice that all the following examples focus on systems with only one layer of boundary condition. Also, notice that E_0 and H_0 does not have to be placed on E_z and H_z; placing on other components work fine too, but the results may look different.

- **1-D Constraint (Box in x-direction) Waveguide:** $\mu\varepsilon\omega^2 - k_z^2 - (\frac{m\pi}{a})^2 = 0$.

	TM mode	TE mode
	$E_z = E_0 \sin\frac{m\pi x}{a} e^{\pm ik_z z}$	$H_z = H_0 \cos\frac{m\pi x}{a} b e^{\pm ik_z z}$
	$E_x = \frac{\pm ik_z a E_0}{m\pi}\cos\frac{m\pi x}{a} e^{\pm ik_z z}$	$H_x = \frac{\pm ik_z a H_0}{m\pi}\sin\frac{m\pi x}{a} e^{\pm ik_z z}$
	$H_y = \frac{i\varepsilon\omega E_0}{m\pi}\cos\frac{m\pi x}{a} e^{\pm ik_z z}$	$E_y = \frac{i\mu\omega H_0}{m\pi}\sin\frac{m\pi x}{a} e^{\pm ik_z z}$
Lowest	$m = 0$ (TEM mode)	if $a > b$, then $m = 1$, $n = 0$

- **2-D Constraint (Box in xy-direction) Waveguide:** $\mu\varepsilon\omega^2 - k_z^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2 = 0$.

	TM mode	TE mode
	$E_z = E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} e^{\pm ik_z z}$	$H_z = H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} e^{\pm ik_z z}$
Lowest	$m = n = 1$	if $a > b$, then $m = 1$, $n = 0$
	$E_z = E_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} e^{\pm ik_z z}$	$H_z = H_0 \cos\frac{\pi x}{a} e^{\pm ik_z z}$
	$\vec{E}_t = \frac{\pm ik_z a^2 b^2 E_0}{\pi(a^2+b^2)}\left(\frac{\hat{x}}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{\hat{y}}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\right) e^{\pm ik_z z}$	$H_x = \frac{\mp ik_z a}{\pi} H_0 \sin\frac{\pi x}{a} e^{\pm ik_z z}$
	$\vec{H}_t = \frac{i\varepsilon\omega a^2 b^2 E_0}{\pi(a^2+b^2)}\left(\frac{\hat{y}}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{\hat{x}}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\right) e^{\pm ik_z z}$	$E_y = \frac{i\mu\omega a}{\pi} H_0 \sin\frac{\pi x}{a} e^{\pm ik_z z}$

- **2-D Constraint (Cylinder) Waveguide:** $\mu\varepsilon\omega^2 - k_z^2 - (\frac{x_{mn}}{a})^2 = 0$.

	TM mode	TE mode
	$J_m(x_{mn}) = 0$	$J'_m(x_{mn}) = 0$
	$E_z = E_0 J_m\left(\frac{x_{mn}}{a}\rho\right) e^{\pm im\phi \pm ik_z z}$	$H_z = H_0 J_m\left(\frac{x_{mn}}{a}\rho\right) e^{\pm im\phi \pm ik_z z}$
Lowest	$x_{01} = 2.405$	$x_{11} = 1.841$
	$E_z = E_0 J_0\left(\frac{2.405}{a}\rho\right) e^{\pm ik_z z}$	$H_z = H_0 J_1\left(\frac{1.841}{a}\rho\right) e^{\pm i\phi \pm ik_z z}$
	$E_\rho = \frac{\mp ik_z a}{2.405} E_0 J_1\left(\frac{2.405}{a}\rho\right) e^{\pm ik_z z}$	$\vec{H}_t = \frac{\pm ik_z a^2}{(1.841)^2} H_0 \left[\frac{1.841}{a} J'_1\left(\frac{1.841}{a}\rho\right)\hat{\rho} \pm \frac{i}{\rho} J_1\left(\frac{2.405}{a}\rho\right)\hat{\phi}\right] e^{\pm i\phi \pm ik_z z}$
	$H_\phi = \frac{-i\varepsilon\omega a}{2.405} E_0 J_1\left(\frac{2.405}{a}\rho\right) e^{\pm ik_z z}$	$\vec{E}_t = \frac{-i\mu\omega a^2}{(1.841)^2} H_0 \left[\frac{1.841}{a} J'_1\left(\frac{2.405}{a}\rho\right)\hat{\phi} \mp \frac{i}{\rho} J_1\left(\frac{2.405}{a}\rho\right)\hat{\rho}\right] e^{\pm i\phi \pm ik_z z}$

- **3-D Constraint (Box) Resonance Cavity:** $\mu\varepsilon\omega^2 - (\frac{p\pi}{L})^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2 = 0.$

TM mode

$$E_z = E_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \cos\frac{p\pi z}{L}$$

Lowest $m = n = 1$, $p = 0$

$$E_z = E_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\vec{H}_t = \frac{i\pi}{\mu\omega}\left(\frac{\hat{y}}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{\hat{x}}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\right)$$

TE mode

$$H_z = H_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \sin\frac{p\pi z}{L}$$

Lowest: if $a > b$, then $m = p = 1$, $n = 0$

$$H_z = H_0 \cos\frac{\pi x}{a} \sin\frac{\pi z}{L}$$

$$H_x = \frac{a}{L} H_0 \sin\frac{\pi x}{a} \cos\frac{\pi z}{L}$$

$$E_y = \frac{-i\mu\omega a}{\pi} H_0 \sin\frac{\pi x}{a} \sin\frac{\pi z}{L}$$

- **3-D Constraint (Cylinder) Resonance Cavity:** $\mu\varepsilon\omega^2 - (\frac{p\pi}{L})^2 - (\frac{x_{mn}}{a})^2 = 0.$

TM mode

$$J_m(x_{mn}) = 0$$

$$E_z = E_0 J_m\left(\frac{x_{mn}}{a}\rho\right) e^{\pm im\phi} \cos\frac{p\pi z}{L}$$

Lowest $x_{01} = 2.405$, $p = 0$

$$E_z = E_0 J_0\left(\frac{2.405}{a}\rho\right)$$

$$H_\phi = -i\sqrt{\frac{\varepsilon}{\mu}} E_0 J_1\left(\frac{1.841}{a}\rho\right)$$

TE mode

$$J'_m(x_{mn}) = 0$$

$$H_z = H_0 J_m\left(\frac{x_{mn}}{a}\rho\right) e^{\pm im\phi} \sin\frac{p\pi z}{L}$$

Lowest $x_{11} = 1.841$, $p = 1$

$$H_z = H_0 J_1\left(\frac{1.841}{a}\rho\right) e^{\pm i\phi} \sin\frac{\pi z}{L}$$

$$\vec{H}_t = \frac{\pi a^2}{L(1.841)^2} H_0 \left[\frac{1.841}{a} J'_1\left(\frac{1.841}{a}\rho\right)\hat{\rho} \pm \frac{i}{\rho} J_1\left(\frac{2.405}{a}\rho\right)\hat{\phi}\right] e^{\pm i\phi} \cos\frac{\pi z}{L}$$

$$\vec{E}_t = \frac{-i\mu\omega a^2}{(1.841)^2} H_0 \left[\frac{1.841}{a} J'_1\left(\frac{2.405}{a}\rho\right)\hat{\phi} \mp \frac{i}{\rho} J_1\left(\frac{2.405}{a}\rho\right)\hat{\rho}\right] e^{\pm i\phi} \sin\frac{\pi z}{L}$$

- **3-D Constraint (Sphere) Resonance Cavity:** choose the radial direction to define TM/TE mode. Solve for the exact solutions of the Helmholtz equation with boundary conditions in spherical coordinate (the solutions are in term of spherical Bessel function; see Jackson problem 10.7 to understand the tricks, even though the question is for scattering) under circular polarization (so as to use raising operator on spherical harmonics).

Electrodynamics: Sourced Radiation

- **Far Zone:** $kr \gg 1$, neglect all $O(\frac{1}{r^2})$ terms; vacuum $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$.

- 3-D Integration by Part:

$$\int \vec{\nabla} \cdot \vec{J} d^3r' = -\int \frac{\vec{J} \cdot \hat{r}}{r'} d^3r', \int \vec{\nabla} \cdot \vec{J}\vec{r}' d^3r' = -\int \vec{J} d^3r', \int \vec{\nabla} \cdot \vec{J} r'_i r'_j d^3r' = -\int (J_i r'_j + J_j r'_i) d^3r'.$$

- Obtain $\vec{J}$:

$$\vec{p} \equiv \int \vec{x}' \rho d^3x', \, \vec{m} \equiv \int \vec{x}' \times \vec{J} d^3x', \, Q_{ij} \equiv \int (3x'_i x'_j - x'^2 \delta_{ij}) \rho d^3x'$$

If dipole $\vec{p}$ is given, then $\rho(\vec{r}', t') = -\vec{p} \cdot \vec{\nabla} \delta(\vec{r}' - \vec{r}_0)$, and by continuity equation,

$$\text{if } \vec{p} = \vec{p}(t') \quad \vec{J}(\vec{r}', t') = \frac{d\vec{p}(t')}{dt'} \delta(\vec{r}' - \vec{r}_0), \quad \text{if } \vec{r}_0 = \vec{v}t' \quad \vec{J}(\vec{r}', t') = -\vec{v}(\vec{p} \cdot \vec{\nabla}) \delta(\vec{r}' - \vec{v}t).$$

If dipole $\vec{m}$ or $\vec{Q} = Q_{\alpha\beta} \hat{r}_\beta$ is given, then replace the $\vec{p}$ in the final result with $\hat{r} \times \vec{m}$ or $\frac{d}{dt} \frac{1}{6c} \vec{Q}$.

- Standard Procedure:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \int dt' \frac{\vec{J}(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \delta\left(t' + \frac{|\vec{r} - \vec{r}'|}{c} - t\right).$$

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}, \, \vec{E} = Z_0 \vec{H} \times \hat{r}.$$

$$\langle \vec{S} \rangle = \int_0^{\frac{2\pi}{\omega}} dt \vec{E} \times \vec{H}^*, \, \frac{dP}{d\Omega} = \text{Re}[r^2 \hat{r} \cdot \vec{S}].$$

- Fourier Transforms: the first can be obtained without $kr \gg 1$ limit,

$$\vec{A}(\vec{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{k|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \vec{J}(\vec{r}', \omega), \, \vec{A}(\vec{k}, t) = \frac{\mu_0}{4\pi r} \int d^3k' \vec{J}\left(\vec{k}', t - \frac{r}{c}\right).$$

- **General:**

- Use Formalism: uses spherical Bessel functions j_l and vector spherical harmonics $\vec{X}_{lm}$.

$$a_E(l, m) = \frac{k^2}{i\sqrt{l(l+1)}} \int d^3x Y^*_{lm} \left[c\rho \frac{\partial}{\partial r} \left(r j_l(kr) \right) + ik(\vec{r} \cdot \vec{J}) j_l(kr) - ik\vec{\nabla} \cdot (\vec{r} \times \vec{M}) j_l(kr) \right],$$

$$a_M(l, m) = \frac{k^2}{i\sqrt{l(l+1)}} \int d^3x Y^*_{lm} \left[\vec{\nabla} \cdot \vec{M} \frac{\partial}{\partial r} \left(r j_l(kr) \right) - k^2 (\vec{r} \cdot \vec{M}) j_l(kr) + \vec{\nabla} \cdot (\vec{r} \times \vec{J}) j_l(kr) \right].$$

With $\vec{L} = \frac{1}{i} \vec{r} \times \vec{\nabla}$ that can be treated as a quantum operator,

$$\vec{X}_{lm}(\theta, \phi) = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm}(\theta.\phi).$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left| \sum_{lm} (-i)^{l+1} [a_E(l, m) \vec{X}_{lm} + a_M(l, m) \hat{r} \times \vec{X}_{lm}] \right|.$$

Scattering

- differential cross section: Z_0 is vacuum impedance for scattering in vacuum, r is the observation distance from source, $\hat{n}$ is observation direction, $\vec{\epsilon}$ is the polarization direction to observe, $\hat{n}_0$ is direction of the incident fields, $\vec{\epsilon}_0$ is the polarization of the incident fields. Note that for isotropic incident wave, the solid angle specifically for $\vec{\epsilon}_0$ needs to be integrated.

$$\frac{d\sigma}{d\Omega} = \frac{r^2 \frac{1}{2Z_0}|\vec{\epsilon}^* \cdot \vec{E}_{\text{sc}}|^2}{\frac{1}{2Z_0}|\vec{\epsilon}^* \cdot \vec{E}_{\text{inc}}|^2}.$$

- **Far Zone:** treat the target as a point source driven in the frequency of the incident.

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_o)^2}\left|\vec{\epsilon}^* \cdot \vec{p} + (\hat{n} \times \vec{\epsilon}^*) \cdot \frac{\vec{m}}{c}\right|.$$

- **Born Approximation:** the k^4 dependence allows an explanation of the blue sky,

- scattering amplitude:

$$\vec{A}_{\text{sc}} = \frac{k^4}{4\pi}\int e^{-ik\vec{n}\cdot\vec{r}'}([\hat{n} \times (\vec{D} - \varepsilon_0\vec{E})] \times \vec{n} - \frac{\varepsilon_0\omega}{k}\vec{n} \times (\vec{B} - \mu_0\vec{H})d^3r', \text{ and } \frac{d\sigma}{d\Omega} = \frac{|\vec{\epsilon}^* \cdot \vec{A}_{\text{sc}}|^2}{|\vec{D}_{\text{inc}}|}.$$

- first order Born approximation: approximation of small change $\delta\epsilon(\vec{r}')$ and $\delta\mu(\vec{r}')$,

$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{\text{sc}}}{|\vec{D}_{\text{inc}}|} = \frac{k^4}{4\pi}\int e^{ik(\vec{n}_0 - \vec{n})\cdot\vec{r}'}\left(\vec{\epsilon}^* \cdot \vec{\epsilon}_0 \frac{\delta\varepsilon(\vec{r}')}{\varepsilon_0} + (\hat{n} \times \vec{\epsilon}^*) \cdot (\hat{n}_0 \times \vec{\epsilon}_0)\frac{\delta\mu(\vec{r}')}{\mu_0}\right)d^3r'.$$

- **General:** refer to spherical resonance cavity. Choices of spherical Bessel functions for scattering can be found according to `http://www.youtube.com/watch?v=zOBqysWUhA8`.

Refraction

- **Kirchhoff Approximation:** basically says use incident fields on the opening/aperture $S_{a'}$; n' is the coordinate perpendicular to the surface of the aperture.

- ψ and $\frac{\partial\psi}{\partial n'}$ are 0 on the boundary other than the aperture.
- ψ and $\frac{\partial\psi}{\partial n'}$ (same k) are continuous on on the aperture.

- **Scalar Kirchhoff:** $\vec{R} \equiv \vec{r} - \vec{r}'$, and $\psi = E_0 e^{-i(\omega t - k\vec{r'}\cdot\hat{n'})}$

$$\psi_{\text{diff}}(\vec{r}) = -\frac{1}{4\pi}\int_{S_{a'}}\frac{e^{ikR}}{R}\hat{n}' \cdot \left[\vec{\nabla}'\psi_{\text{inc}}(\vec{r}') + ik(1 + \frac{i}{kR})\frac{\vec{R}}{R}\psi\right]da'$$

- **Vector Smythe-Kirchhoff:**

$$\vec{E}_{\text{diff}}(\vec{r}) = \frac{1}{2\pi}\vec{\nabla} \times \int_{S_{a'}}(\hat{n}' \times \vec{E}_{\text{inc}}(\vec{r}'))\frac{e^{ikR}}{R}da'$$

- far zone:

$$\vec{E}_{\text{diff}}(\vec{r}) = \frac{ie^{ikr}}{2\pi}\vec{k} \times \int_{S_{a'}}(\hat{n}' \times \vec{E}_{\text{inc}}(\vec{r}'))e^{i\vec{k}\cdot\vec{r}'}da'$$

- Note about the Babinet's principle.

Relativistic Electrodynamics

- convention: suppose S and $\bar{S}$ reference frames,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \text{ boost in z-direction } \Lambda^\mu_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}.$$

- **4-vectors and Their Invariance:**

- definition of vector field vx: $v^\mu(x^\nu) \to \bar{v}^\mu(\bar{x}^\nu) = \Lambda^\mu_\alpha v^\alpha(x^\nu) = \Lambda^\mu_\alpha v^\alpha(x^\nu(\bar{x})) = \Lambda^\mu_\alpha v^\alpha((\Lambda^\nu_\beta)^{-1}\bar{x}^\beta)$. - differential proper time: $d\tau = \frac{dt}{\gamma}$.

Name	Definition	Invariance
4-position	$x^\mu = (ct, x, y, z)$	$x_\mu x^\mu = x^2 = (c\tau)^2$
4-velocity	$u^\mu = (c, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau})$	$u_\mu u^\mu = c^2$
4-acceleration	$a^\mu = (0, \frac{du_x}{d\tau}, \frac{du_y}{d\tau}, \frac{du_z}{d\tau})$	$u_\mu a^\mu = 0$
4-gradient	$\partial^\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z})$	$\partial_\mu \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}\right)^2 - \vec{\nabla}$
4-momentum	$p^\mu = (\frac{1}{c}E, m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau},)$	$p_\mu p^\mu = (mc)^2$
4-current	$j^\mu = (c\rho, j_x, j_y, j_z)$	$\partial_\mu j^\mu = 0$
4-potential	$A^\mu = (\phi, A_x, A_y, A_z)$	$\partial_\mu A^\mu = 0.$

$$\partial_\alpha \partial^\alpha A^\mu = \frac{4\pi}{c} j^\mu.$$

- **E&M field basics:** check Weinberg,

- field tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}, \ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

-Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu, \ \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0.$$

- Lorentz force law:

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} u_\nu.$$

- **Retarded Source:** a indicates the particle number, and z_a is ath particle's location,

$$j^\mu(x) = \sum_a q_a c \int u^\mu_a(\tau_a) \delta^4(x - z_a(\tau_a)) d\tau_a.$$

- mechanical stress tensor:

$$T^{\mu\nu}_{\text{mech}}(x) = \sum_a m_a c \int u^\mu_a(\tau_a) u^\nu_a(\tau_a) \delta^4(x - z_a(\tau_a)) d\tau_a.$$

- **Liénard-Wiechert Potentials:** suppose single electron, and $\dot{\vec{\beta}} \equiv \frac{d\vec{\beta}}{dt}$,

$$A^\mu(x) = 2e\int u^\mu(\tau)\Theta(x^0 - z'^0(\tau))\delta[(x - z(\tau))^2]d\tau = \frac{eu^\mu(\tau)}{u\cdot(x - z(\tau))}\bigg|_{\tau=\tau_0}$$

- fields: $(R, R\vec{n}) \equiv (x - z)^\mu$,

$$\vec{E}(x) = e\left[\frac{\vec{n} - \vec{\beta}}{\gamma^2(1 - \vec{\beta}\cdot\vec{n})R^2}\right]_{\text{ret}} + \frac{e}{c}\left[\frac{\vec{n}\times[(\vec{n} - \vec{\beta})\times\dot{\vec{\beta}}]}{(1 - \vec{\beta}\cdot\vec{n})^3 R}\right]_{\text{ret}}, \ \vec{B} = [\vec{n}\times\vec{E}]_{\text{ret}}.$$

- power: $t' \equiv t - \frac{R(t')}{c}$ is the retarded time,

$$P = -\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) = \frac{3}{2}\frac{e^2}{c}\gamma^6[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2],$$

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{|\vec{n}\times[(\vec{n} - \vec{\beta})\times\dot{\vec{\beta}})]|^2}{(1 - \vec{n}\cdot\vec{b})^5} = \underbrace{\frac{e^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}}_{\vec{\beta}\parallel\dot{\vec{\beta}}}, \quad \underbrace{\frac{e^2\dot{v}^2}{4\pi c^3}\left[\frac{1}{(1 - \beta\cos\theta)^3} - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\cos\theta)^5}\right]}_{\vec{\beta}\perp\dot{\vec{\beta}}}.$$

- if the motion of an electron has a fix period T, $\omega_0 \equiv \frac{2\pi}{T}$, its radiation has discrete mode m,

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}\left|\int_0^T \vec{t}\times\vec{n}e^{im\omega_0\left(t - \frac{\vec{n}\cdot\vec{x}(t)}{c}\right)}dt\right|^2$$

- frequency dependence of energy spectrum: $\epsilon^\mu(\hat{k}) \equiv (0, \epsilon_1, \epsilon_2, \hat{k})$,

$$\frac{dE}{d\omega d\Omega} = \frac{\omega^2}{2c^5(2\pi)^3}|\epsilon^{\mu*}(\hat{k})^\mu j(\vec{k})_\mu|^2.$$

- Note about Cherenkov radiation.