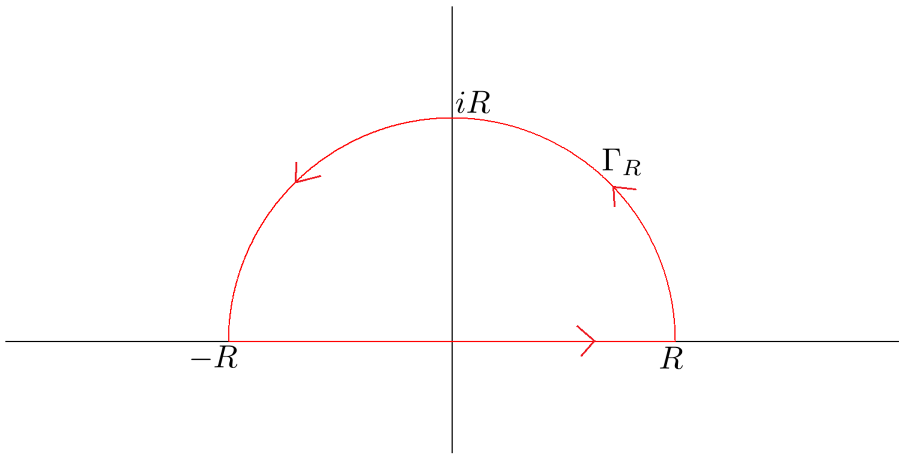
Calculate the integral

$$\int_0^\infty \frac{\cos(2x)}{x^2+2}\ \mathrm{d}x$$

Solution: First note that the integrand is even, hence

$$\int_0^\infty \frac{\cos(2x)}{x^2+2}\ \mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(2x)}{x^2+2}\ \mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\Re\left(e^{2ix}\right)}{x^2+2}\ \mathrm{d}x$$

Integrate $f(z) = \frac{e^{2iz}}{z^2+2}$ along the curve Γ_R



The function f has the single pole $\sqrt{2}i$ inside Γ_R (given $R > \sqrt{2}$), and

$$\operatorname{res}\{f, \sqrt{2}i\} = \lim_{z\to\sqrt{2}i} \frac{e^{2iz}}{z^2+2}\left(z-\sqrt{2}i\right) = \lim_{z\to\sqrt{2}i} \frac{e^{2iz}}{z+\sqrt{2}i} = \frac{e^{-2\sqrt{2}}}{2\sqrt{2}i}$$

hence

$$\int_{\Gamma_R} f\ \mathrm{d}z = \frac{\pi e^{-2\sqrt{2}}}{\sqrt{2}}$$

By Jordan, the integral along $\Gamma_R \backslash [-R, R]$ $\left(\text{that is, the integral along } \left\{Re^{i\theta} \mid \theta \in (0,\pi)\right\}\right)$ equals 0, and since $e^{2iz} = \cos(2z) + i\sin(2z)$ where $\sin(2z)$ is odd, we see that

$$\int_{[-R,R]} \frac{\cos(2z)}{z^2+2}\ \mathrm{d}z = \frac{\pi e^{-2\sqrt{2}}}{\sqrt{2}} - i\underbrace{\int_{[-R,R]} \frac{\sin(2z)}{z^2+2}\ \mathrm{d}z}_{=0}$$

That is,

$$\int_0^\infty \frac{\cos(2x)}{x^2+2}\ \mathrm{d}x = \frac{\pi e^{-2\sqrt{2}}}{2\sqrt{2}}$$