THE PRIME HARMONIC SERIES DIVERGES

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Abstract. We present a modern proof that the sum of the reciprocals of the primes diverges.

In 1737 Leonhard Euler presented a paper entitled *Variae observationes circa series infinitas (Various observations about infinite series)* to the St. Petersburg Academy [Eul44], in which, among other results, he proved that the sum of the reciprocals of primes diverges.[1] This proof would not be considered rigorous by modern standards, but is regarded as essentially correct, in that its ideas can be made rigorous.

For most students it is at first a surprising fact that infinite sums whose terms converge to 0, such as the harmonic series, may still diverge. The following proof is a folklore classic.

Proposition 1. *The harmonic series* $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ *diverges.*

Proof. Suppose that the harmonic series converges to some finite number H. Then

$$\begin{aligned} H &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \cdots \\ &\geq 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{8} + \tfrac{1}{8} + \cdots && \text{(since } \tfrac{1}{2k-1} + \tfrac{1}{2k} \geq \tfrac{1}{2k} + \tfrac{1}{2k}\text{)} \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots && \text{(since } \tfrac{1}{2k} + \tfrac{1}{2k} = \tfrac{1}{k}\text{)} \\ &= \tfrac{1}{2} + H, \end{aligned}$$

which is a contradiction. □

Though the harmonic series diverges, it is not difficult using the integral test for series to see that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for every $s > 1$ (see, for example, ([Spi06], §23, Thm. 4)). Euler himself explicitly evaluated this sum when $n = 2$, showing that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (see [Eul37]), thereby solving the so-called *Basel problem.* Given that the harmonic series corresponds to $s = 1$ in the preceding example, we might be led to believe that it is in some sense the slowest diverging series. However this is far from true.

Let $\mathbb{P}$ denote the set of primes $\{2, 3, 5, 7, 11, \dots\}$. Our aim is to prove the following theorem.

Theorem 2. *The sum of the reciprocals of the primes,* $\sum_{p \in \mathbb{P}} \frac{1}{p}$*, diverges.*

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[1] Euler says that the sum of the series is "$\log\log\infty$"; we would say that he showed that the sum of the first n terms of the series is approximately $\log\log n$.

We'll prove this theorem by means of a formal version of Euler's product formula for the harmonic series:

$$\sum_{n\in\mathbb{N}_+}\frac{1}{n}=\prod_{p\in\mathbb{P}}\frac{1}{1-p^{-1}}=\prod_{p\in\mathbb{P}}\frac{p}{p-1}.$$

Let $\mathcal{P}(\mathbb{N}_+)$ denote the power set of the positive natural numbers. We define a function[2] $\mu\colon \mathcal{P}(\mathbb{N}_+)\to[0,\infty]$ by

$$\mu(X)=\begin{cases}0 & \text{if } X=\varnothing,\\ \sum_{n\in X}\frac{1}{n} & \text{if this sum converges,}\\ \infty & \text{otherwise.}\end{cases}$$

Examples 3.

- Obviously $\mu(X)$ is finite for any finite set X.
- We saw in Proposition 1 that the harmonic series diverges, so $\mu(\mathbb{N}_+)=\infty$.
- For $n\neq 1$, let $[n]$ denote the set $\{1,n,n^2,n^3,n^4,\dots\}$ of powers of n. By the standard evaluation of geometric series, we see that $\mu([n])=\frac{n}{n-1}$.
- The solution of the Basel problem shows that $\mu(\{1,4,9,16,25,\dots\})=\frac{\pi^2}{6}$.
- If $Y\subseteq X$, then $\mu(Y)\leq\mu(X)$.

We now examine the behaviour of μ with respect to a certain transformation T of $\mathcal{P}(\mathbb{N}_+)$. The map $T\colon \mathcal{P}(\mathbb{N}_+)\to\mathcal{P}(\mathbb{N}_+)$, sends X in $\mathcal{P}(\mathbb{N}_+)$ to the set of all finite products of elements of X. That is:

$$T(X)=\left\{x_1^{k_1}x_2^{k_2}\cdots x_r^{k_r}\mid x_i\in X,\ k_i\in\mathbb{N}\right\}.$$

Examples 4.

- $T(\{1\})=\{1\}$.
- For $n\neq 1$ we have $T(\{n\})=[n]$.
- $T(\{2,3\})=\{1,2,3,4,6,8,9,12,15,16,\dots\}$
- Most importantly, $T(\mathbb{P})=\mathbb{N}_+$, by the fundamental theorem of arithmetic.

Theorem 2 is a simple consequence of the following lemma.

Lemma 5. *If $\mu(X)$ is finite, then so is $\mu(T(X))$.*

Proof. We first observe that the result is simple for one-element sets:

$$\mu(T(\{n\}))=\mu([n])=\begin{cases}\frac{n}{n-1} & \text{if } n\neq 1,\\ 1 & \text{if } n=1.\end{cases}$$

Since $\mu(X)$ is finite if and only if $\mu(X\cup\{1\})$ is finite, we may assume $1\notin X$. Suppose that $\mu(T(Y))$ is finite for a given Y. Then we claim that $\mu(T(Y\cup\{n\}))$ is also finite for any $n\in\mathbb{N}_+$. This is vacuous when $n\in Y$, so consider $n\notin Y$. The elements of $T(Y\cup\{n\})$

[2]By $[0,\infty]$ we mean $\{x\in\mathbb{R}|x>0\}\cup\{\infty\}$. Technically speaking, μ is a *measure* on the discrete measure space $(\mathbb{N}_+,\mathcal{P}(\mathbb{N}_+))$.

are all of the form yn^k for $y \in T(Y)$ and $k \in \mathbb{N}$. Given that n may share common factors with elements of $T(Y)$, we see that[3]

$$\sum_{x\in T(Y\cup\{n\})} \frac{1}{x} \leq \sum_{y\in T(Y),k\in\mathbb{N}} \frac{1}{y}\frac{1}{n^k} = \left(\sum_{y\in T(Y)} \frac{1}{y}\right)\cdot\left(\sum_{k\in\mathbb{N}} \frac{1}{n^k}\right) = \left(\sum_{y\in T(Y)} \frac{1}{y}\right)\cdot\frac{n}{n-1}.$$

That is, $\mu(T(Y\cup\{n\})) \leq \mu(T(Y))\cdot\frac{n}{n-1}$. But any subset of $\mathbb{N}_+$ is countable, so writing $X = \{n_1, n_2, n_3, \dots\}$ and iterating the above we have

$$\mu(T(X)) \leq \prod_{i=1}^{\infty} \frac{n_i}{n_i - 1}.$$

To show that $\mu(T(X))$ is finite, it suffices therefore to show that this (possibly infinite) product converges. The convergence of an infinite product $\prod_i a_i$ is equivalent[4] to the convergence of the sum $\sum_i \log(a_i)$, so we only need show that $\sum_{i=1}^{\infty} \log\left(\frac{n_i}{n_i-1}\right)$ is finite. We have $\log(t) < t-1$ for all $t > 1$, so

$$\log\left(\frac{n_i}{n_i-1}\right) < \frac{n_i}{n_i-1} - 1 = \frac{1}{n_i-1} < \frac{2}{n_i}$$

(as $n_i \geq 2$). Therefore we have

$$\sum_{n_i\in X} \log\left(\frac{n_i}{n_i-1}\right) \leq \sum_{n_i\in X} \frac{2}{n_i} = 2\cdot\mu(X).$$

Since $\mu(X)$ was assumed to be finite, the product converges and so the lemma is proved. □

It is now straightforward to see how the divergence of the sum of the reciprocals of the primes follows from the divergence of the harmonic series.

Proof of Theorem 2. We have seen that $T(\mathbb{P}) = \mathbb{N}_+$. By Lemma 5, if $\mu(\mathbb{P})$ were finite, then $\mu(\mathbb{N}_+)$ would be finite as well, contradicting Proposition 1. Hence $\mu(\mathbb{P}) = \infty$ and $\sum_{p\in\mathbb{P}} \frac{1}{p}$ diverges. □

References

[Eul37] Leonhard Euler, *De summis serierum reciprocarum*, Commentarii academiae scientiarum Petropolitanae **7** (1737), 123–134.

[Eul44] ———, *Variae observationes circa series infinitas*, Commentarii academiae scientiarum Petropolitanae **9** (1744), 160–188.

[Spi06] David Spivak, *Calculus*, 3 ed., Cambridge University Press, Cambridge, 2006.

[3] All of our series have positive terms, so they are absolutely convergent whenever they converge. This allows us to use the equality $\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$ (see [Spi06], §24, Thm. 9), and to rearrange convergent series without affecting the sum.

[4] Provided that all terms are greater than 1, see the exercises for ([Spi06], §24).