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1970), 2nd ed., p. 404.
[6]The partition function sum over all configurations of $r(\tau)$ can be cast in the language of an integral over all paths taken by a quantum particle. For the latter approach to quantum mechanics, see, e.g., R. P. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[7]B. D. Sullivan and M. M. Telo da Gama, in *Fluid Interfacial Phenomena*, edited by C. A. Croxton (Wiley, New York, 1986).

Supersymmetry, shape invariance, and exactly solvable potentials

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It is well known that the harmonic oscillator potential can be solved by using raising and lowering operators. This operator method can be generalized with the help of supersymmetry and the concept of "shape-invariant" potentials. This generalization allows one to calculate the energy eigenvalues and eigenfunctions of essentially all known exactly solvable potentials in a simple and elegant manner.

I. INTRODUCTION

Most textbooks on nonrelativistic quantum mechanics show how the harmonic oscillator potential can be elegantly solved by the raising and lowering operator method.[1] The purpose of this article is to describe a generalization of the operator method[2] that can be used to handle many more potentials of physical interest. The generalization is based on two main concepts: supersymmetry and shape invariant potentials. For quantum-mechanical purposes, the main implication of supersymmetry is simply stated. Given any potential $V_-(x)$, supersymmetry allows one to construct a partner potential $V_+(x)$ with the same energy eigenvalues (except for the ground state).[3,4] Furthermore, if $V_-(x)$ and $V_+(x)$ have similar shapes, they are said to be "shape invariant." This concept was introduced three years ago by Gendenshtein.[5] He calculated the energy eigenvalue spectrum and pointed out that essentially all known solvable potentials[6] (Coulomb, harmonic oscillator, Morse, Eckart, Pöschl–Teller, etc.) are shape invariant.[7] This work has been extended by us [2] to a calculation of all the bound state wavefunctions from the ground state in a manner analogous to the harmonic oscillator operator method.

The whole development is very elegant, appealing, and yet rather simple, so that any student of quantum mechanics should be able to understand and appreciate it. Indeed, we strongly feel that the material presented here can be profitably included in future quantum mechanics courses and textbooks. Accordingly, we have kept this article at a pedagogical level and made it as self-contained as possible. In Sec. II, we give a quick review of the standard operator method for solving the one-dimensional simple harmonic oscillator potential in nonrelativistic quantum mechanics. Section III contains a summary of the main ideas of supersymmetric quantum mechanics. Section IV is the heart of this article. In it, we precisely define the notion of shape invariant potentials, and then show how one can simply obtain the energy eigenvalues [Eq. (36)] and eigenfunctions [Eqs. (47) and (48)] by a generalized operator method. A useful table of all known shape-invariant potentials and their eigenstates is given. A discussion of related problems and concluding remarks are contained in Sec. V.

II. OPERATOR METHOD FOR THE HARMONIC OSCILLATOR

The one-dimensional harmonic oscillator Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2. \tag{1}$$

In terms of the raising and lowering operators a^+ and a defined by

$$a = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\frac{d}{dx} + \frac{1}{2}\left(\frac{2m\omega}{\hbar}\right)^{1/2}x,$$
$$a^+ = -\left(\frac{\hbar}{2m\omega}\right)^{1/2}\frac{d}{dx} + \frac{1}{2}\left(\frac{2m\omega}{\hbar}\right)^{1/2}x, \tag{2}$$

the Hamiltonian takes the form

$$H = (a^+a + \tfrac{1}{2})\,\hbar\omega. \tag{3}$$

The following commutation relations are easily derived.

$$[a,a^+] = 1, \quad [a,H] = a\hbar\omega, \quad [a^+,H] = -a^+\hbar\omega. \tag{4}$$

The utility of operators a and a^+ comes from their ability to generate new eigenstates from a given one. In particular, if ψ_n is an eigenfunction of H with eigenvalue E_n, then $a\psi_n$ and $a^+\psi_n$ are also eigenfunctions with eigenvalues $E_n - \hbar\omega$ and $E_n + \hbar\omega$, respectively. Since the operator a^+a in H is positive semidefinite, all eigenvalues $E_n \geqslant \frac{1}{2}\hbar\omega$. Therefore, the successive lowering of eigenstates by the operator a must eventually stop at the ground-state wavefunction ψ_0, by requiring

$$a\psi_0(x) = 0. \tag{5}$$

Operating with $\hbar\omega a^+$ yields

$$\hbar\omega a^+a\psi_0(x) = (H - \tfrac{1}{2}\hbar\omega)\psi_0(x) = 0, \tag{6}$$

which corresponds to a ground-state energy $E_0 = \frac{1}{2}\hbar\omega$. Also, using the definition of the lowering operator a [Eq.

(2)] makes Eq. (5) a simple differential equation to solve. This gives the normalized ground-state wavefunction

$$\psi_0(x) = (m\omega/\pi\hbar)^{1/4} \exp[-(m\omega/2\hbar)x^2]. \quad (7)$$

All higher eigenstates are generated by repeated applications of a^+. The complete energy spectrum is

$$E_n = (n+\tfrac{1}{2})\hbar\omega, \quad n = 0,1,2,..., \quad (8)$$

with corresponding normalized eigenfunctions

$$\psi_n(x) = [(a^+)^n/\sqrt{n!}]\ \psi_0(x). \quad (9)$$

The operator method is clearly fast, and it bypasses the lengthy solution of the Schrödinger differential equation in terms of Hermite polynomials. Of course, if desired, these explicit eigenfunctions can also be obtained from Eqs. (9) and (2).[1]

III. SUPERSYMMETRIC QUANTUM MECHANICS

Assume that one has a potential $V_-(x)$ whose ground-state wavefunction $\psi_0^{(-)}(x) \equiv \psi_0(x)$ is known, and whose ground-state energy has been adjusted so that $E_0^{(-)} = 0$.[8] Then the Schrödinger equation for the ground state is

$$H_-\psi_0 = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_-(x)\right)\psi_0 = 0, \quad (10)$$

and consequently,

$$H_- = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0}\right). \quad (11)$$

Define the operators

$$A^+ = \frac{\hbar}{\sqrt{2m}}\left(-\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right);$$

$$A = \frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right). \quad (12)$$

This gives

$$A^+A = H_-, \quad AA^+ \equiv H_+ = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_+(x), \quad (13)$$

where

$$V_+(x) = V_-(x) - \frac{\hbar^2}{m}\frac{d}{dx}\left(\frac{\psi_0'}{\psi_0}\right) = -V_-(x) + \frac{\hbar^2}{m}\left(\frac{\psi_0'}{\psi_0}\right)^2. \quad (14)$$

The potentials V_+ and V_- are called supersymmetric partner potentials. As we shall shortly show, they have the same energy levels (except for $E_0^{(-)} = 0$). Also note from Eq. (12) that $A\psi_0 = 0$.

In supersymmetric quantum mechanics,[4] it is customary to use the superpotential $W(x)$ rather than the ground-state eigenfunction ψ_0 of H_-. Here, $W(x)$ and ψ_0 are related by

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\left(\frac{\psi_0'}{\psi_0}\right);$$

$$\psi_0(x) = \exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x W(x)dx\right). \quad (15)$$

In terms of the superpotential $W(x)$, the operators A and A^+ are given by

$$A^+ = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x),$$

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x). \quad (16)$$

Also, the supersymmetric partner potentials $V_\pm(x)$ are given by

$$V_\pm(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2m}}W'(x); \quad W'(x) = \frac{dW}{dx}. \quad (17)$$

Note that $W^2(x)$ is the average of the potentials $V_+(x)$ and $V_-(x)$, whereas $W'(x)$ is proportional to the commutator of A and A^+[9]:

$$\tfrac{1}{2}[V_+(x) + V_-(x)] = W^2(x);$$

$$[A,A^+] = (2\hbar/\sqrt{2m})W'(x). \quad (18)$$

Let $\psi_n^{(-)}$ and $\psi_n^{(+)}$ denote the eigenfunctions of the Hamiltonians H_- and H_+, respectively, with eigenvalues $E_n^{(-)}$ and $E_n^{(+)}$. The integer $n = 0,1,2,3,...,$ denotes the number of nodes in the wavefunction. We will now show that the potentials V_+ and V_- have the same energy spectrum, except that the ground-state energy $E_0 = 0$ of V_- has no corresponding level for V_+. More specifically, if $\psi_n^{(-)}$ is any eigenfunction of H_- with eigenvalue $E_n^{(-)}$, then $A\psi_n^{(-)}$ is an eigenfunction of H_+ with the same eigenvalue. The proof is straightforward.

$$H_+(A\psi_n^{(-)}) = AA^+(A\psi_n^{(-)}) = AH_-\psi_n^{(-)} = E_n^{(-)}(A\psi_n^{(-)}). \quad (19)$$

Similarly, if $\psi_n^{(+)}$ is an eigenfunction of H_+ with eigenvalue $E_n^{(+)}$, then $A^+\psi_n^{(+)}$ is an eigenfunction of H_- with the same eigenvalue:

$$H_-(A^+\psi_n^{(+)}) = A^+A(A^+\psi_n^{(+)}) = A^+H_+\psi_n^{(+)} = E_n^{(+)}(A^+\psi_n^{(+)}). \quad (20)$$

From Eqs. (19) and (20), it is clear that the eigenstates of H_+ are simply related to those of H_-:

$$E_n^{(+)} = E_{n+1}^{(-)}, \quad (21)$$

$$\psi_n^{(+)} = [E_{n+1}^{(-)}]^{-1/2}A\psi_{n+1}^{(-)} \quad (n = 0,1,2,3,...). \quad (22)$$

If the eigenfunction $\psi_{n+1}^{(-)}$ of H_- is normalized, then the wavefunction $\psi_n^{(+)}$ of H_+ in Eq. (22) is also normalized. Note that the operator A not only converts an eigenfunction of H_- into an eigenfunction of H_+ with the same energy, but it also destroys a node [$\psi_{n+1}^{(-)}$ has $(n+1)$ nodes, whereas $\psi_n^{(+)}$ has n]. Similarly, one has

$$\psi_{n+1}^{(-)} = [E_n^{(+)}]^{-1/2}A^+\psi_n^{(+)} \quad (n = 0,1,2,3,...). (23)$$

The operator A^+ creates a node and converts an eigenfunction of H_+ into an eigenfunction of H_- with the same energy.

In Fig. 1(a), we show the energy spectra of two supersymmetric partner potentials V_- and V_+. The roles played by the operators A and A^+ are indicated: They connect states of the same energy for two different (supersymmetric partner) potentials. For comparison, in Fig. 1(b), we show the one-dimensional harmonic oscillator spectrum.

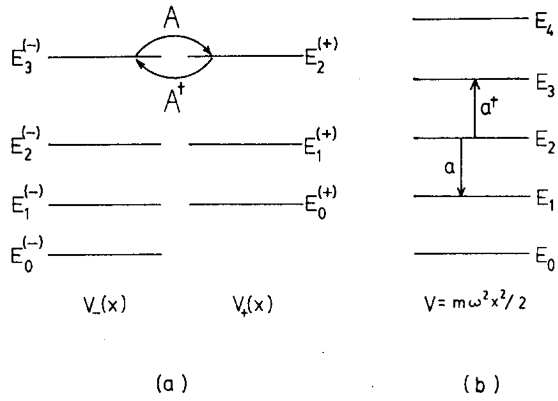

Fig. 1. (a) The eigenvalue spectra of the supersymmetric partner potentials $V_-(x)$ and $V_+(x)$. The action of the operators A and A^+ is indicated. (b) The equally spaced energy levels of the one-dimensional simple harmonic oscillator. The action of the raising and lowering operators a^+ and a is shown.

The roles played by the operators a and a^+ are indicated: They connect states of different energies for the same (harmonic oscillator) potential.

An explicit example will clarify the ideas presented in this section. Consider a particle of mass m in an infinite square well of width L.

$$V(x) = \begin{cases} 0 & (0 \leqslant x \leqslant L), \\ \infty & (-\infty < x < 0, \quad x > L). \end{cases} \tag{24}$$

The eigenstates are well known[1]:

$$E_n = \frac{(n+1)^2 h^2}{8mL^2}, \qquad \psi_n = \left(\frac{2}{L}\right)^{1/2} \sin\frac{(n+1)\pi x}{L} \quad (0 \leqslant x \leqslant L), \tag{25}$$

where $n = 0,1,2,3,...,$ is the number of nodes in the wavefunction in the interval $0 < x < L$. Clearly, the shifted potential

$$V_-(x) = V(x) - h^2/8mL^2 \tag{26}$$

will have a ground-state energy $E_0^{(-)} = 0$. The eigenstates of $V_-(x)$ are

$$E_n^{(-)} = \frac{n(n+2)h^2}{8mL^2}, \qquad \psi_n^{(-)} = \left(\frac{2}{L}\right)^{1/2} \sin\frac{(n+1)\pi x}{L} \quad (0 \leqslant x \leqslant L). \tag{27}$$

From Eq. (15), the superpotential is

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\pi}{L} \cot\left(\frac{\pi x}{L}\right) \quad (0 \leqslant x \leqslant L). \tag{28}$$

Using either Eq. (14) or (17), it is now easy to find the supersymmetric partner potential $V_+(x)$ of the infinite square well $V_-(x)$.

$$V_+(x) = (h^2/8mL^2)\,[2\csc^2(\pi x/L) - 1]. \tag{29}$$

The wavefunctions of $V_+(x)$ are obtained from Eqs. (22) and (27).

$$\psi_n^{(+)} \propto \left[\frac{d}{dx} - \frac{\pi}{L}\cot\left(\frac{\pi x}{L}\right)\right] \sin\frac{(n+2)\pi x}{L} \propto (n+1)\cos\frac{(n+2)\pi x}{L} - \sin\frac{(n+1)\pi x}{L} \Big/ \sin\frac{\pi x}{L}. \tag{30}$$

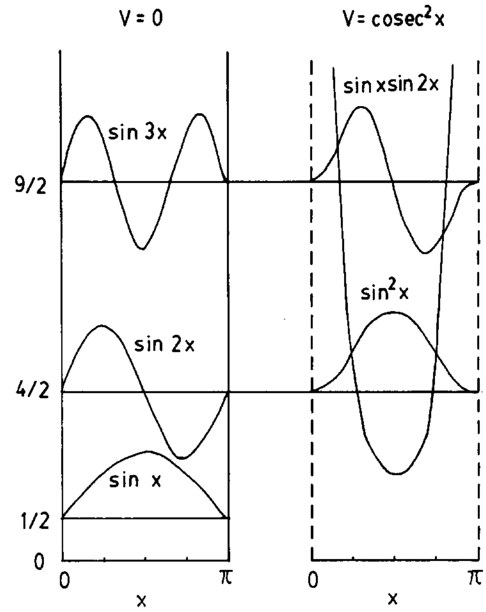

Fig. 2. The low-lying energy eigenstates of the infinite square well of width π and its supersymmetric partner potential $\csc^2 x$. The units used are $\hbar = m = 1$.

In particular,

$$\psi_0^{(+)} \propto \sin^2\left(\frac{\pi x}{L}\right), \quad \psi_1^{(+)} \propto \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right). \tag{31}$$

Thus we have obtained the nice result that the familiar infinite square well $V(x)$ [Eq. (24)] and the potential $(h^2/4mL^2)\csc^2(\pi x/L)$ have exactly the same energy levels except for the ground state. These potentials and their low-lying wavefunctions are shown in Fig. 2, where the well width has been chosen to be $L = \pi$ and $\hbar = m = 1$.

IV. SHAPE INVARIANT POTENTIALS AND THEIR EIGENSTATES

There are a number of analytically solvable problems in nonrelativistic quantum mechanics for which all the energy eigenvalues and eigenfunctions are explicitly known.[6] Some examples are the Coulomb, harmonic oscillator, Morse, Eckart, and Pöschl–Teller potentials. The question naturally arises as to why these potentials are solvable. What is the underlying symmetry property, if any? No unified answer was known until about 3 years ago, when in a largely unnoticed paper, Gendenshtein[5] pointed out that all these potentials have the property of "shape invariance."

Let us now explain precisely what one means by shape invariant potentials. If the pair of supersymmetric partner potentials $V_\pm(x)$ defined by Eq. (17) are similar in shape and differ only in the parameters that appear in them, then they are said to be shape invariant. More specifically, if $V_-(x;a_0)$ is any potential, its supersymmetric partner $V_+(x;a_0)$ must satisfy the requirement[5]

$$V_+(x;a_0) = V_-(x;a_1) + R(a_1), \tag{32}$$

where a_0 is a set of parameters, a_1 is a function of a_0 [$a_1 = f(a_0)$ say], and the remainder $R(a_1)$ is independent of x.

We now show that the eigenstates of shape invariant potentials can be easily obtained. To that purpose, construct a series of Hamiltonians $H^{(s)}, s=0,1,2,\ldots$, where $H^{(0)} \equiv H_-$, $H^{(1)} \equiv H_+$.

$$H^{(s)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_-(x;a_s) + \sum_{k=1}^{s} R(a_k), \qquad (33)$$

where $a_s = f^s(a_0)$, i.e., the function f applied s times. Let us compare the spectrum of $H^{(s)}$ with that of $H^{(s+1)}$. In view of Eqs. (32) and (33), we have

$$H^{(s+1)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_-(x;a_{s+1}) + \sum_{k=1}^{s+1} R(a_k)$$
$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_+(x;a_s) + \sum_{k=1}^{s} R(a_k). \quad (34)$$

Comparing Eqs. (33) and (34), we see that $H^{(s)}$ and $H^{(s+1)}$ are supersymmetric partner Hamiltonians and hence have identical bound-state energy spectra except for the lowest level of $H^{(s)}$ whose energy is

$$E_0^{(s)} = \sum_{k=1}^{s} R(a_k). \qquad (35)$$

This follows from Eq. (33) and the fact that $E_0^{(-)} = 0$. On going back from $H^{(s)}$ to $H^{(s-1)}$, we would eventually reach $H^{(1)}(=H_+)$ and $H^{(0)}(=H_-)$, whose ground-state energy is zero and its nth energy level being coincident with the ground state of Hamiltonian $H^{(n)}$ $(n=1,2,3,\ldots)$. Hence, the complete energy spectrum of H_- is given by[5]

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k), \quad E_0^{(-)} = 0. \qquad (36)$$

As a simple, physically interesting example, let us consider the potential

$$V(x) = -V_0 \operatorname{sech}^2 \beta x. \qquad (37)$$

Potentials of this shape can be generated from the superpotential

$$W(x) = A \tanh \beta x, \quad A > 0. \qquad (38)$$

In fact, using Eq. (17), the supersymmetric partner potentials are

$$V_-(x;A) = A^2 - A(A + \beta\hbar/\sqrt{2m})\operatorname{sech}^2 \beta x;$$
$$V_+(x;A) = A^2 - A(A - \beta\hbar/\sqrt{2m})\operatorname{sech}^2 \beta x. \qquad (39)$$

Clearly, one can write

$$V_+(x;A) = V_-(x;A - \beta\hbar/\sqrt{2m}) + A^2 - (A - \beta\hbar/\sqrt{2m})^2, \qquad (40)$$

which is precisely the requirement for shape invariance. In fact, comparing Eqs. (40) and (32) yields

$$a_0 = A; \quad a_1 = f(a_0) = A - \beta\hbar/\sqrt{2m};$$
$$R(a_1) = a_0^2 - a_1^2. \qquad (41)$$

Clearly, successive Hamiltonians in the sequence $H^{(s)}$ $(s=0,1,2,\ldots)$ will have parameters $a_s = f^s(a_0) = A - s\beta\hbar/\sqrt{2m}$. Therefore, the bound-state energies of the potential $V_-(x;A) \equiv V_-(x;a_0)$ are

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k) = \sum_{k=1}^{n} (a_{k-1}^2 - a_k^2)$$
$$= a_0^2 - a_n^2 = A^2 - (A - n\beta\hbar/\sqrt{2m})^2. \qquad (42)$$

The ground-state wavefunction of $V_-(x;a_0)$ is calculated from Eq. (15):

$$\psi_0^{(-)}(x;a_0) \propto (\operatorname{sech} \beta x)^s; \quad s \equiv \sqrt{2m}A/\hbar\beta. \qquad (43)$$

The requirement $A > 0$ [that we have assumed in Eq. (38)] guarantees that $\psi_0^{(-)}(x;a_0)$ is normalizable.

The energy levels E_n of the potential $-V_0 \operatorname{sech}^2 \beta x$ [Eq. (37)] can be obtained from those of $V_-(x;a_0)$ [Eq. (39)] by subtracting A^2 from $E_n^{(-)}$ and identifying

$$V_0 = A(A + \beta\hbar/\sqrt{2m}). \qquad (44)$$

Solving for A and requiring $A > 0$ gives

$$A = -\frac{\beta\hbar}{2\sqrt{2m}} + \frac{1}{2}\sqrt{\frac{\beta^2\hbar^2}{2m} + 4V_0}. \qquad (45)$$

Therefore, the energy levels of the potential $V(x) = -V_0 \operatorname{sech}^2 \beta x$ are

$$E_n = E_n^{(-)} - A^2$$
$$= -\left(A - \frac{n\beta\hbar}{\sqrt{2m}}\right)^2$$
$$= -\frac{\hbar^2\beta^2}{8m}\left(-(1+2n) + \sqrt{1 + \frac{8mV_0}{\beta^2\hbar^2}}\right)^2, \qquad (46)$$

which is well known to be the correct answer.[1c]

In Table I, we give a list of all known shape invariant potentials $V_-(x;a_0)$ and their bound-state energy spectra $E_n^{(-)}$. For each case, we also give the superpotential $W(x)$ and values of the parameters a_0, a_1, and the remainder $R(a_1)$. All the familiar analytically solvable potentials[6,7] correspond to special values of the parameters in one of the cases given in Table I.

We now show that for any shape invariant potential $V_-(x;a_0)$, the bound-state wavefunctions $\psi_n^{(-)}(x;a_0)$ can be easily constructed from the ground-state wavefunction $\psi_0^{(-)}(x;a_0)$ [which is known from the superpotential via Eq. (15)]. This is possible since the operators A and A^+ link up the eigenfunctions of the same energy for supersymmetric partner Hamiltonians H_+ and H_-. Let us start from the Hamiltonian $H^{(s)}$ as given by Eq. (33). Its ground-state eigenfunction is given by $\psi_0^{(-)}(x;a_s)$. On going from $H^{(s)}$ to $H^{(s-1)}$ to $H^{(1)}(=H_+)$ and $H^{(0)}(=H_-)$ and using Eq. (23) we then find that the nth state unnormalized energy eigenfunction $\psi_n^{(-)}(x;a_0)$ for the original Hamiltonian $H_-(x;a_0)$ is given by

$$\psi_n^{(-)}(x;a_0) \propto A^+(x;a_0)A^+(x;a_1)\cdots A^+(x;a_{n-1})\psi_0^{(-)}(x;a_n), \qquad (47)$$

which is clearly a generalization of the operator method of constructing the energy eigenfunctions for the harmonic oscillator problem discussed in Sec. II. In fact it is now clear why in the oscillator problem the operators a^+ and a can be interpreted as energy raising and lowering operators. The point is that the oscillator potential is shape invariant with $a_0 = a_1 = a_2 = a_n = \omega$ so that the ground-state eigenfunctions of the class of Hamiltonians $H^{(s)}$ $(s=0,1,2,\ldots)$ are all the same. In this case, Eq. (47) reduces to Eq. (9), since the operator A^+ defined in Eq. (16) becomes proportional to the raising operator a^+ of Eq. (2).

In practice, if one wants explicit expressions for the wavefunctions, it is simpler to use the result

$$\psi_{n+1}^{(-)}(x;a_0) \propto A^+(x;a_0)\psi_n^{(-)}(x;a_1), \qquad (48)$$

rather than the equivalent Eq. (47). Equation (48) follows from Eq. (23) when one notes that for shape-invariant po-

Table I. All known shape invariant potentials and their properties are given. Unless otherwise specified, the range of these potentials is $-\infty < x < \infty$, $0 \leqslant r < \infty$.

Name of potential	Superpotential $W(x)$	Potential $V_-(x;a_0)$	$\{a_0\}$	$\{a_1\}$	$R(a_1)$	Eigenvalues $E_n^{(-)}$	Ground-state wavefunction $\psi_0^{(-)}$
Shifted oscillator	$\sqrt{\frac{m}{2}}\omega x - b$	$\frac{1}{2}m\omega^2\left(x-\sqrt{\frac{2}{m}}\frac{b}{\omega}\right)^2 - \frac{\hbar\omega}{2}$	ω	ω	$\hbar\omega$	$n\hbar\omega$	$\exp\left[-\frac{m\omega}{2\hbar}\left(x-\sqrt{\frac{2}{m}}\frac{b}{\omega}\right)^2\right]$
Three-dimensional oscillator	$\sqrt{\frac{m}{2}}\omega r - \frac{(l+1)\hbar}{\sqrt{2m}\,r}$	$\frac{1}{2}m\omega^2 r^2 + \frac{l(l+1)\hbar^2}{2mr^2} - (l+\frac{3}{2})\hbar\omega$	l	$l+1$	$2\hbar\omega$	$2n\hbar\omega$	$r^{l+1}\exp\left(-\frac{m\omega r^2}{2\hbar}\right)$
Coulomb	$\sqrt{\frac{m}{2}}\frac{e^2}{(l+1)\hbar} - \frac{(l+1)\hbar}{\sqrt{2m}\,r}$	$-\frac{e^2}{r} + \frac{l(l+1)\hbar^2}{2mr^2} - \frac{me^4}{2(l+1)^2\hbar^2}$	l	$l+1$	$\frac{me^4}{2\hbar^2}\left[\frac{1}{(a_0+1)^2} - \frac{1}{(a_1+1)^2}\right]$	$\frac{me^4}{2\hbar^2}\left[\frac{1}{(l+1)^2} - \frac{1}{(n+l+1)^2}\right]$	$r^{l+1}\exp\left(-\frac{me^2 r}{(l+1)\hbar^2}\right)$
Morse	$A - Be^{-\alpha x}$	$A^2 + B^2 e^{-2\alpha x} - 2B\left(A+\frac{\alpha\hbar}{2\sqrt{2m}}\right)e^{-\alpha x}$	A	$A - \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$\exp\left[-\frac{\sqrt{2m}}{\hbar}\left(Ax+\frac{B}{\alpha}e^{-\alpha x}\right)\right]$
	$A\tanh\alpha x + B\,\mathrm{sech}\,\alpha x$	$A^2 + \left(B^2 - A^2 - \frac{A\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}^2\alpha x + B\left(2A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}\,\alpha x\tanh\alpha x$	A	$A - \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$(\mathrm{sech}\,\alpha x)^{\sqrt{2m}A/\alpha\hbar} \times\exp\left[-\frac{2\sqrt{2m}B}{\alpha\hbar}\tan^{-1}(e^{\alpha x})\right]$
Rosen–Morse	$A\tanh\alpha x + \frac{B}{A}$	$A^2 + \frac{B^2}{A^2} + 2B\tanh\alpha x - A\left(A+\frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}^2\alpha x$	A	$A - \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2 + B^2\left[\frac{1}{a_0^2} - \frac{1}{a_1^2}\right]$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 + B^2\left[\frac{1}{A^2} - \frac{1}{\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2}\right]$	$(\mathrm{sech}\,\alpha x)^{\sqrt{2m}A/\alpha\hbar} \times\exp\left(-\frac{\sqrt{2m}Bx}{\hbar A}\right)$
	$A\coth\alpha r - B\,\mathrm{csch}\,\alpha r$ $(A < B)$	$A^2 + \left(B^2 + A^2 + \frac{A\alpha\hbar}{\sqrt{2m}}\right)\mathrm{csch}^2\alpha r - B\left(2A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\coth\alpha r\,\mathrm{csch}\,\alpha r$	A	$A - \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$\frac{(\sinh\alpha r)^{(\sqrt{2m}/\alpha\hbar)(B-A)}}{(1+\cosh\alpha r)^{\sqrt{2m}B/\alpha\hbar}}$
Eckart	$-A\coth\alpha r + \frac{B}{A}$ $(B > A^2)$	$A^2 + \frac{B^2}{A^2} - 2B\coth\alpha r + A\left(A - \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{csch}^2\alpha r$	A	$A + \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2 + B^2\left[\frac{1}{a_0^2} - \frac{1}{a_1^2}\right]$	$A^2 - \left(A + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 + B^2\left[\frac{1}{A^2} - \frac{1}{\left(A+\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2}\right]$	$(\sinh\alpha r)^{\sqrt{2m}A/\alpha\hbar} \times\exp\left(-\frac{\sqrt{2m}Br}{\hbar A}\right)$
	$-A\cot\alpha x + B\csc\alpha x$ $(0 \leqslant \alpha x \leqslant \pi;\ A > B)$	$-A^2 + \left(A^2 + B^2 - \frac{A\alpha\hbar}{\sqrt{2m}}\right)\csc^2\alpha x - B\left(2A - \frac{\alpha\hbar}{\sqrt{2m}}\right)\cot\alpha x\csc\alpha x$	A	$A + \frac{\alpha\hbar}{\sqrt{2m}}$	$a_1^2 - a_0^2$	$\left(A + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 - A^2$	$(\sin\alpha x)^{\sqrt{2m}(A-B)/\alpha\hbar}$ $(1+\cos\alpha x)^{\sqrt{2m}B/\alpha\hbar}$
Pöschl–Teller I	$A\tan\alpha x - B\cot\alpha x$ $(0 \leqslant \alpha x \leqslant \pi/2)$	$-(A+B)^2 + A\left(A - \frac{\alpha\hbar}{\sqrt{2m}}\right)\sec^2\alpha x + B\left(B - \frac{\alpha\hbar}{\sqrt{2m}}\right)\csc^2\alpha x$	(A,B)	$\left(A + \frac{\alpha\hbar}{\sqrt{2m}},\ B + \frac{\alpha\hbar}{\sqrt{2m}}\right)$	$\left(A + B + \frac{2\alpha\hbar}{\sqrt{2m}}\right)^2 - (A+B)^2$	$\left(A + B + \frac{2n\alpha\hbar}{\sqrt{2m}}\right)^2 - (A+B)^2$	$(\sin\alpha x)^{\sqrt{2m}B/\alpha\hbar} \times(\cos\alpha x)^{\sqrt{2m}A/\alpha\hbar}$
Pöschl–Teller II	$A\tanh\alpha r - B\coth\alpha r$ $(B < A)$	$(A-B)^2 - A\left(A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}^2\alpha r + B\left(B - \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{csch}^2\alpha r$	(A,B)	$\left(A - \frac{\alpha\hbar}{\sqrt{2m}},\ B + \frac{\alpha\hbar}{\sqrt{2m}}\right)$	$(A-B)^2 - \left(A - B - \frac{2\alpha\hbar}{\sqrt{2m}}\right)^2$	$(A-B)^2 - \left(A - B - \frac{2n\alpha\hbar}{\sqrt{2m}}\right)^2$	$\frac{(\sinh\alpha r)^{\sqrt{2m}B/\alpha\hbar}}{(\cosh\alpha r)^{\sqrt{2m}A/\alpha\hbar}}$

tentials

$$\psi_n^{(+)}(x;a_0) = \psi_n^{(-)}(x;a_1). \qquad (49)$$

Repeated application of Eq. (48) for $n = 0,1,2,3,\ldots$, gives all the eigenfunctions. The procedure for successively obtaining higher-energy eigenfunctions stops if any wavefunction is not normalizable. Of course, this corresponds to the case, where a potential can only hold a finite number of bound states.

Note that for $A^+(x;a_0)$ in Eq. (48), one can either use Eq. (16) in terms of the superpotential $W(x)$ or alternatively use Eq. (12) in terms of the ground-state wavefunction $\psi_0 \equiv \psi_0^{(-)}(x;a_0)$. If the latter choice is made, one has yet another useful expression for the eigenfunctions.

$$\psi_{n+1}^{(-)}(x;a_0) \propto \frac{1}{\psi_0}\frac{d}{dx}\left[\psi_0\psi_n^{(-)}(x;a_1)\right]. \qquad (50)$$

Again, as an illustration, we compute the low-lying wavefunctions of the potential $-V_0 \operatorname{sech}^2 \beta x$ [Eq. (37)]. The ground-state wavefunction $\psi_0^{(-)}(x;a_0)$ was already computed [Eq. (43)]. The first excited eigenfunction is

$$\psi_1^{(-)}(x;a_0) \propto A^+(x;a_0)\psi_0^{(-)}(x;a_1) = \left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + A\tanh\beta x\right)(\operatorname{sech}\beta x)^{s-1} \propto \tanh\beta x(\operatorname{sech}\beta x)^{s-1}. \quad (51)$$

Similarly, the second excited state wavefunction is

$$\psi_2^{(-)}(x;a_0) \propto A^+(x;a_0)\psi_1^{(-)}(x;a_1) = \left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + A\tanh\beta x\right) \times \tanh\beta x(\operatorname{sech}\beta x)^{s-2} \propto [-1 + (2s-1)\tanh^2\beta x](\operatorname{sech}\beta x)^{s-2}. \quad (52)$$

These results agree with the known analytic solutions.[1c] For any given value of s, the number of bound states is the smallest integer greater than s. The ground-state wavefunctions for all known shape invariant potentials are also given in Table I.

V. CONCLUSIONS

The ideas of supersymmetric quantum mechanics have many applications. Among these have been the study of atomic systems,[10] evaluation of the eigenvalues of a bistable potential,[11] improvement of large N expansions,[12] and development of a more accurate WKB approximation.[2,13] In this article we have focused on the problem of analytically solvable potentials. We have shown that the operator method of solving the harmonic oscillator can be generalized to all shape invariant potentials—a class that includes many solvable problems of physical interest.[7] Table I contains a list of all known shape invariant potentials and properties of their eigenstates. It should be noted that although the terminology and ideas are quite different, the techniques of supersymmetry when applied to quantum mechanics are essentially equivalent to the method of factorization of the Hamiltonian.[3] Indeed, the mathematical condition necessary for the method of factorization to work corresponds to the physical condition of shape invariance [Eq. (32)] of supersymmetric partner potentials. Basically, both methods are special cases of an old procedure developed by Darboux for handling second-order linear differential equations.[14] It would be interesting to know if more solutions of the shape invariance condition exist.[7] In any case, it is clear that supersymmetry and shape invariance give us a simple, elegant, deeper understanding of analytically solvable potentials.

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