Expected number of tosses to get x heads in a row with a coin that lands heads with probability p.

OK, so what we are interested in is a number of tosses E after which we can expect a sequence of x heads in a row, given that the specific coin we are tossing lands on heads with a probability p. We will end with an equation of the form:

$$E_{x,p} = f(p, x)$$

To get a better intuition for this problem I'm going to think about a specific case first and slowly make it general. Let's say we have a fair coin $p = \frac{1}{2}$ and we're trying to get 5 heads in a row. We can get a valid equation involving the expected value just by considering some conditions for success or failure.

Sequence of flips encountered	Change in expected number of flips, E
T	Keep flipping: $E + 1$
HT	Keep flipping: $E + 2$
HHT	Keep flipping: $E + 3$
HHHT	Keep flipping: $E + 4$
HHHHT	Keep flipping: $E + 5$
HHHHH	Done! $E =$ current number of flips

Because each coin toss is independent, whenever we get a sequence where we fail like T, we have the same exact chance of flipping heads 5 times in a row again *after* that flip. So our expected number of flips *after* flipping T is still E, but remembering we just made a flip, we can expect to get 5 heads in $E + 1$ flips total. We can extend this thinking to all of the failing coin flip sequences.

For HHHHH, we know how many coin flips are needed to get 5 in a row, because we have just flipped them—$E = 5$.

Now, we can think about how often each of these possibilities occur, and use it to get an expression that we can then move around to get E in terms of p and x.

Probability of happening	Seq. of flips encountered	Total num. of flips needed
$\frac{1}{2}$	T	$E+1$
$\frac{1}{4}$	HT	$E+2$
$\frac{1}{8}$	HHT	$E+3$
$\frac{1}{16}$	HHHT	$E+4$
$\frac{1}{32}$	HHHHT	$E+5$
$\frac{1}{32}$	HHHHH	5

Now, we can add up all of these expected numbers of flips multiplied by their probability of occurring to give us our total E.

$$E = \tfrac{1}{2}(E+1) + \tfrac{1}{4}(E+2) + \tfrac{1}{8}(E+3) + \tfrac{1}{16}(E+4) + \tfrac{1}{32}(E+5) + \tfrac{1}{32}(5)$$

Then it's just algebra the rest of the way home.

$$E = \frac{1}{2}(E+1) + \frac{1}{4}(E+2) + \frac{1}{8}(E+3) + \frac{1}{16}(E+4) + \frac{1}{32}(E+5) + \frac{1}{32}(5)$$

$$E = \frac{31}{32}E + \frac{62}{32}$$

$$E - \frac{31}{32}E = \frac{62}{32}$$

$$E(1 - \frac{31}{32}) = \frac{62}{32}$$

$$E(\frac{1}{32}) = \frac{62}{32}$$

$$E = 62$$

So now we have a little bit of background that will help us think of the more general case. How could we generalize the equation we got for E for any probability p and any number x of consecutive heads?

Thinking about the sequence of tosses that got us each of the probabilities for the fair coin will help us to find all of the likelihoods of a certain sequence happening for a coin that has chance p of landing on heads. If p is the chance we get heads, then $(1-p)$ is the chance we get tails. So, finding the probability of any sequence is as easy as substituting those probabilities in for the specific sequence of tosses we want. For example, the probability we get HHHT would be $p * p * p * (1-p)$. Extending that principle we can say:

Probability of happening	Seq. of flips encountered	Total num. of flips needed
$(1-p)$	T	$E+1$
$p(1-p)$	HT	$E+2$
$p^2(1-p)$	HHT	$E+3$
$p^3(1-p)$	HHHT	$E+4$
$p^4(1-p)$	HHHHT	$E+5$
p^5	HHHHH	5

Alright, we've generalized for any sort of coin weighted to land on heads or tails more frequently. Let's see what the formula would look like in this instance (using summation notation to simplify the terms with E in them).

$$E = \sum_{k=1}^{5} p^{k-1}(1-p)(E+k) + 5p^5$$

All that's left now is to see how the formula would change when we define success as any arbitrary length of heads x. For any length we would need to account for all the failing sequences from T through $\underbrace{\text{HHH}\ldots\text{HT}}_{x}$, and lastly the sequence we defined as success $\underbrace{\text{HHH}\ldots\text{H}}_{x}$.

The probability of the sequence happening and the number of flips you need for success when each of the sequences occurs both follow directly from these general length sequences:

Probability of happening	Seq. of flips encountered	Total num. of flips needed
$(1-p)$	T	$E+1$
$p(1-p)$	HT	$E+2$
$p^2(1-p)$	HHT	$E+3$
$\dots$	$\dots$	$\dots$
$p^{x-1}(1-p)$	HHH ... HT	$E+x$
p^x	HHH ... HH	x

Finally we're left with:

$$E = \sum_{k=1}^{x} p^{k-1}(1-p)(E+k) + xp^x$$

All that's left now is more algebra and a few tricks involving summation.

$$E = xp^x + \sum_{k=1}^{x} p^{k-1}(1-p)(E+k)$$
$$E = xp^x + (1-p)\sum_{k=1}^{x} p^{k-1}(E+k)$$
$$E = xp^x + (1-p)\left(\sum_{k=1}^{x} p^{k-1}E + \sum_{k=1}^{x} p^{k-1}k\right)$$
$$E = xp^x + (1-p)\left(\frac{E}{p}\sum_{k=1}^{x} p^{k} + \sum_{k=1}^{x} p^{k-1}k\right)$$

At this junction, I use the fact that:

$$\sum_{k=1}^{x} p^k = \frac{1-p^{x+1}}{1-p} - 1 \text{ and that } \sum_{k=1}^{x} p^{k-1}k = \frac{1-p^x}{(1-p)^2} - \frac{xp^x}{(1-p)}$$

So we are left with:

$$E = xp^x + (1-p)\left(\frac{E}{p}\left(\frac{1-p^{x-1}}{1-p} - 1\right) + \left(\frac{1-p^x}{(1-p)^2} - \frac{xp^x}{(1-p)}\right)\right)$$

$$E = xp^x + \left(\frac{E}{p}\left((1-p^{x+1}) - (1-p)\right) + \left(\frac{1-p^x}{(1-p)} - xp^x\right)\right)$$

$$E = xp^x + \left(\frac{E}{p}\left(p - p^{x+1}\right) + \frac{1-p^x}{(1-p)} - xp^x\right)$$

$$E = E(1-p^x) + \frac{1-p^x}{(1-p)}$$

$$E - E(1-p^x) = \frac{1-p^x}{(1-p)}$$

$$E(1-(1-p^x)) = \frac{1-p^x}{(1-p)}$$

$$E(p^x) = \frac{1-p^x}{(1-p)}$$

And so we finish with our expected length in terms of p and x:

$$E_x = \frac{1-p^x}{p^x(1-p)}$$

From a problem I got from Math Stack Exchange:

http://math.stackexchange.com/questions/364038/expected-number-of-coin-tosses-to-get-five-consecutive-heads

-/u/janoseye