CHAPTER 6

Adjoint functors

1. Adjoint pairs of functors

Definition 6.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. An *adjunction* between F and G is an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, G(-))$$

of functors $\mathcal{C}^{\mathrm{opp}} \times \mathcal{D} \to \mathbf{Set}$. If such an adjunction exists, we say that F is *left adjoint* to G, and that G is *right adjoint* to F.

In other words, an adjunction from F to G consists of the data of a bijection

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, GY), \tag{7}$$

for every X in $\mathcal{C}$ and Y in $\mathcal{D}$, such that for every $f\colon X_1 \to X_2$ the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX_1, Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(X_1, GY) \\ {\scriptstyle -\circ Ff}\uparrow & & \uparrow{\scriptstyle -\circ f} \\ \mathrm{Hom}_{\mathcal{D}}(FX_2, Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(X_2, GY) \end{array}$$

commutes, and for every $g\colon Y_1 \to Y_2$ the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX, Y_1) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(X, GY_1) \\ \downarrow{\scriptstyle g\circ -} & & \downarrow{\scriptstyle Gg\circ -} \\ \mathrm{Hom}_{\mathcal{D}}(FX, Y_2) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(X, GY_2) \end{array}$$

commutes. Informally, we say that (7) is a 'functorial isomorphism' or an 'isomorphism, functorial in X and Y'. In practice, the adjunction (7) tends to be a canonical bijection whose functoriality is clear from the definition. We then sometimes just write

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) = \mathrm{Hom}_{\mathcal{C}}(X, GY),$$

following a common abuse of notation to write *canonical* isomorphisms using the = symbol.

Remark 6.2. The terminology comes from an analogy with linear algebra: if V and W are vector spaces equipped with inner products, then linear maps $f\colon V \to W$ and $g\colon W \to V$ are called adjoint if we have

$$\langle f(v), w\rangle_W = \langle v, g(w)\rangle_V$$

for all $v \in V$ and $w \in W$.

Assume that $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint of $G\colon \mathcal{D} \to \mathcal{C}$, with an adjunction

$$\operatorname{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(X, GY). \tag{8}$$

Taking $Y = FX$, we find

$$\operatorname{Hom}_{\mathcal{D}}(FX, FX) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(X, GFX),$$

and hence the map id_{FX} in $\mathcal{D}$ corresponds to a canonical map

$$\eta_X\colon X \to GFX,$$

which by the functoriality of the adjunction (8) defines a morphism of functors

$$\eta\colon \mathrm{id}_{\mathcal{C}} \to GF.$$

Similarly, taking $X = GY$ in (8) we obtain a morphism of functors

$$\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}.$$

The morphisms of functors η and ϵ are called the *unit* and *co-unit* of the adjunction between F and G.

2. Many examples

The main reason that adjunctions between functors are interesting, is that they are ubiquitous: they arise surprisingly often in multiple branches of mathematics. Here is a short list of examples.

Example 6.3 (Cartesian product and set of maps)**.** If X, Y and A are sets, then we have a canonical bijection

$$\operatorname{Hom}(X \times A, Y) = \operatorname{Hom}(X, \operatorname{Hom}(A, Y))$$

given from left to right by mapping a function $f\colon X \times A \to Y$ to the function

$$X \to \operatorname{Hom}(A, Y),\ x \mapsto (a \mapsto f(x, a))$$

and from right to left by mapping a function $g\colon X \to \mathrm{Hom}(A, Y)$ to

$$X \times A \to Y,\ (x, a) \mapsto g(x)(a).$$

In other words, if we fix a set A, then the functor

$$\mathbf{Set} \to \mathbf{Set},\ X \mapsto X \times A$$

is left adjoint to the functor

$$\mathbf{Set} \to \mathbf{Set},\ Y \mapsto \mathrm{Hom}(A, Y).$$

The unit $\eta\colon \mathrm{id} \to \mathrm{Hom}(A, - \times A)$ of this adjunction is given by

$$\eta_X\colon X \to \mathrm{Hom}(A, X \times A),\ x \mapsto (a \mapsto (x, a))$$

and the co-unit $\epsilon\colon \mathrm{Hom}(A, -) \times A \to \mathrm{id}$ is given by

$$\epsilon_X\colon \mathrm{Hom}(A, X) \times A \to X,\ (f, a) \mapsto f(a).$$

Example 6.4 (Tensor product and Hom)**.** This is a variation on the previous example. Let R be a commutative ring and let A be an R-module. Then the functor

$$\mathbf{Mod}_R \to \mathbf{Mod}_R,\ M \mapsto M \otimes_R A$$

is left adjoint to the functor

$$\mathbf{Mod}_R \to \mathbf{Mod}_R,\ M \mapsto \mathrm{Hom}_R(A, M),$$

which comes down to the functorial isomorphism

$$\mathrm{Hom}_R(M \otimes_R A,\ N) \xrightarrow{\sim} \mathrm{Hom}_R(M,\ \mathrm{Hom}_R(A, N))$$

of Theorem 5.12.

Example 6.5 (Free module and forgetful functor)**.** Let R be a ring. Let M be an R-module and let $R^{(I)}$ be the free R-module on a set I (see Example 4.6). Then we claim that there is a canonical bijection

$$\mathrm{Hom}_{\mathbf{Mod}_R}(R^{(I)}, M) = \mathrm{Hom}_{\mathbf{Set}}(I, M).$$

Indeed, a module homomorphism $R^{(I)} \to M$ is uniquely determined by the images of the basis vectors δ_i, and conversely, given a map of sets $f\colon I \to M$ we obtain an R-module homomorphism

$$R^{(I)} \to M,\ \varphi \mapsto \sum_{i \in I} \varphi(i) f(i).$$

This is just a reformulation of the familiar fact from linear algebra: to give a linear map from V to W is the same as to give the images of the vectors in a basis of V.

But now, if we denote by

$$G\colon \mathbf{Mod}_R \to \mathbf{Set},\ M \mapsto M$$

the forgetful functor (see Example 4.3) and by

$$F\colon \mathbf{Set} \to \mathbf{Mod}_R,\ I \mapsto R^{(I)}$$

be the free module functor then we conclude that we have functorial isomorphisms

$$\mathrm{Hom}_{\mathbf{Mod}_R}(FI, M) = \mathrm{Hom}_{\mathbf{Set}}(I, GM)$$

and hence that the free module functor F is a left adjoint to the forgetful functor G. The unit of this adjunction is the morphism $\eta\colon \mathrm{id} \to GF$ given by the function

$$\eta_I\colon I \to R^{(I)},\ i \mapsto \delta_i,$$

for every set I.

Example 6.6 (Discrete topology, forgetful functor, trivial topology)**.** Any function *from* a discrete topological space is automatically continuous. Likewise, any function *to* a trivial topological space is automatically continuous. That is, we have

$$\mathrm{Hom}_{\mathbf{Top}}(X_{\mathrm{d}isc}, Y) = \mathrm{Hom}_{\mathbf{Set}}(X, Y)$$

and

$$\mathrm{Hom}_{\mathbf{Set}}(X, Y) = \mathrm{Hom}_{\mathbf{Top}}(X, Y_{\mathrm{t}riv}),$$

and we see that the discrete topology functor

$$\mathbf{Set} \to \mathbf{Top},\ X \mapsto X_{\mathrm{d}isc}$$

is left adjoint to the forgetful functor $\mathbf{Top} \to \mathbf{Set}$, and that the trivial topology functor

$$\mathbf{Set} \to \mathbf{Top},\ Y \mapsto Y_{\mathrm{t}riv}$$

is right adjoint to the forgetful functor.

Example 6.7 (Frobenius reciprocity)**.** Let k be a field, let G be a group and let $H \subset G$ be a subgroup. Then Frobenius reciprocity gives for every k-linear representation V of H and W of G a canonical isomorphism

$$\mathrm{Hom}_{k[G]}(\mathrm{Ind}_H^G V,\ W) = \mathrm{Hom}_{k[H]}(V,\ \mathrm{Res}_H^G W),$$

hence the functor Ind_H^G is a left adjoint to Res_H^G.

3. Yoneda and uniqueness of adjoints

If X is an object in $\mathcal{C}$, then we have a functor

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X)\colon \mathcal{C}^{\mathrm{opp}} \to \mathbf{Set}.$$

It maps an object Y to $\mathrm{Hom}(Y, X)$, and if $f\colon Y_1 \to Y_2$ is a morphism in $\mathcal{C}$, then we obtain a morphism of sets

$$h_X(f)\colon\ \mathrm{Hom}_{\mathcal{C}}(Y_2, X) \to \mathrm{Hom}_{\mathcal{C}}(Y_1, X),\ g \mapsto gf.$$

Now the functors from $\mathcal{C}$ to $\mathbf{Set}$ form themselves a category $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$, in which the morphisms are the natural transformations between functors. The above construction defines a functor

$$h\colon \mathcal{C} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathbf{Set}),\ X \mapsto h_X$$

On the level of morphisms it is given by sending a map $f\colon X \to Y$ to the natural transformation $h_f\colon h_X \to h_Y$ given by

$$h_{f,T}\colon\ \mathrm{Hom}_{\mathcal{C}}(T, X) \to \mathrm{Hom}_{\mathcal{C}}(T, Y),\ g \mapsto fg$$

for every T in $\mathcal{C}$.

Theorem 6.8 (Yoneda's Lemma)**.** *The functor*

$$\mathcal{C} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathbf{Set}),\ X \mapsto h_X$$

is fully faithful.

Proof. In other words, we need to show that for all pairs of objects X, Y in $\mathcal{C}$ the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathbf{Set})}(h_X, h_Y) \tag{9}$$

is a bijection. It is easy to define a map in the other way: Let $\varphi\colon h_X \to h_Y$ be a morphism of functors. Then for every T we have a map $\varphi_T\colon h_X(T) \to h_Y(T)$, and in particular, taking $T = X$, we find a map

$$\varphi_X\colon\ \mathrm{Hom}_{\mathcal{C}}(X, X) \to \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

The image of id_X now gives a canonical element

$$f := \varphi_X(\mathrm{id}_X) \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

a morphism $f\colon X \to Y$ in $\mathcal{C}$.

To see that this construction is a two-sided inverse to (9), note that for every $g\colon T \to X$ the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,X) & \xrightarrow{f\circ -} & \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ \big\downarrow{\scriptstyle -\circ g} & & \big\downarrow{\scriptstyle -\circ g} \\ \mathrm{Hom}_{\mathcal{C}}(T,X) & \xrightarrow{f\circ -} & \mathrm{Hom}_{\mathcal{C}}(T,Y) \end{array}$$

commutes. Tracing the element id_X we find

$$\begin{array}{ccc} \mathrm{id}_X & \xmapsto{f\circ -} & \varphi_X(\mathrm{id}_X) \\ \big\downarrow{\scriptstyle -\circ g} & & \big\downarrow{\scriptstyle -\circ g} \\ g & \xmapsto{f\circ -} & \varphi_T(g) \end{array}$$

and hence that $\varphi_T(g)$ is completely determined by $f := \varphi_X(\mathrm{id}_X)$. □

Corollary 6.9. *If h_X and h_Y are isomorphic functors, then X and Y are isomorphic objects in $\mathcal{C}$.* □

Corollary 6.10 (Uniqueness of right adjoints)**.** *If both $G_1\colon \mathcal{D} \to \mathcal{C}$ and $G_2\colon \mathcal{D} \to \mathcal{C}$ are right adjoints to a functor $F\colon \mathcal{C} \to \mathcal{D}$, then G_1 and G_2 are isomorphic functors.*

Proof. Choose adjunctions between F and G_1 and between F and G_2. Then we obtain isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, G_1Y) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G_2Y),$$

functorial in X and Y. Functoriality in X implies that for every Y in $\mathcal{D}$ we find an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(-, G_1Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, G_2Y)$$

of functors $\mathcal{C}^{\mathrm{opp}} \to \mathbf{Set}$ and hence by Yoneda's lemma (Theorem 6.8) an isomorphism

$$\alpha_Y\colon G_1Y \xrightarrow{\sim} G_2Y$$

in $\mathcal{C}$. Functoriality in Y implies that the collection $(\alpha_Y)_Y$ defines an isomorphism of functors

$$\alpha\colon G_1 \xrightarrow{\sim} G_2$$

which finishes the proof. □

There is (of course) a dual of Yoneda's lemma. Given an object X in $\mathcal{C}$, consider the functor

$$h^X := \mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathbf{Set}.$$

We have a functor

$$\mathcal{C}^{\mathrm{opp}} \to \mathbf{Fun}(\mathcal{C}, \mathbf{Set}),\ X \mapsto h^X$$

On the level of morphisms it is given by sending a map $f\colon X \to Y$ to the natural transformation $h^f\colon h^Y \to h^X$ given by

$$h^f_T\colon\ \mathrm{Hom}_{\mathcal{C}}(Y, T) \to \mathrm{Hom}_{\mathcal{C}}(X, T),\ g \mapsto gf$$

for every T in $\mathcal{C}$.

Theorem 6.11 (co-Yoneda's Lemma). *The functor*

$$\mathcal{C}^{\mathrm{opp}} \to \mathbf{Fun}(\mathcal{C}, \mathbf{Set}),\ X \mapsto h^X$$

is fully faithful. □

Corollary 6.12 (Uniqueness of left adjoints). *If both $F_1\colon \mathcal{C} \to \mathcal{D}$ and $F_2\colon \mathcal{C} \to \mathcal{D}$ are left adjoints to a functor $G\colon \mathcal{D} \to \mathcal{C}$, then F_1 and F_2 are isomorphic functors.* □

Exercises

Exercise 6.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence with quasi-inverse $G\colon \mathcal{D} \to \mathcal{C}$. Show that F is both left and right adjoint to G.

Exercise 6.2. Show that the abelianization functor $G \mapsto G^{\mathrm{ab}}$ (see Example 4.5) is a left adjoint to the inclusion functor $\mathbf{Ab} \to \mathbf{Grp}$. What are the unit and co-unit of this adjunction?

Exercise 6.3. Let $\mathcal{R}$ be the category with $\operatorname{ob}\mathcal{R} = \mathbf{R}$ and

$$\operatorname{Hom}_{\mathcal{R}}(x, y) = \begin{cases} \{\star\} & x \leq y \\ \emptyset & x > y \end{cases}$$

for all $x, y \in \mathbf{R}$ (see also Example 3.9). Let $\mathcal{Z}$ be the full sub-category with $\operatorname{ob}\mathcal{Z} = \mathbf{Z}$ and let $F\colon \mathcal{Z} \to \mathcal{R}$ be the inclusion functor. Does this functor have a left adjoint? And a right adjoint?

Exercise 6.4. Assume $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is left adjoint to $G\colon \mathcal{C}_2 \to \mathcal{C}_1$ and $F'\colon \mathcal{C}_2 \to \mathcal{C}_3$ is left adjoint to $G'\colon \mathcal{C}_3 \to \mathcal{C}_2$. Show that $F'F$ is left adjoint to GG'.

Exercise 6.5. Let $\{\star\}$ be the 'one-point category' consisting of a unique object $\star$ and a unique morphism $\mathrm{id}_\star$. Let $\mathcal{C}$ be an arbitrary category. When does the (unique) functor $\mathcal{C} \to \{\star\}$ have a left adjoint? And a right adjoint?

Exercise 6.6. For a set I denote by $\mathbf{Z}[X_i \mid i \in I]$ the polynomial ring in variables (X_i) indexed by I. Elements of $\mathbf{Z}[X_i \mid i \in I]$ are finite $\mathbf{Z}$-linear combinations of monomials in finitely many of the variables. Verify that $I \mapsto \mathbf{Z}[X_i \mid i \in I]$ defines a functor $\mathbf{Set} \to \mathbf{CRing}$ which is left adjoint to the forgetful functor $\mathbf{CRing} \to \mathbf{Set}$.

Exercise 6.7. Show that the forgetful functor $\mathbf{Top}_\star \to \mathbf{Top}$ has a left adjoint but not a right adjoint.

Exercise 6.8. Look up the definition of Stone-Čech compactification, and verify that it gives a left adjoint to the inclusion functor from the category of compact Hausdorff spaces to $\mathbf{Top}$.

Exercise 6.9 (Triangle identities ($\star$))**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$, with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and co-unit $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$.

Show that the diagrams

$$\begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FGFX \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \epsilon_{FX}} \\ & & FX \end{array} \qquad \qquad \begin{array}{ccc} GY & \xrightarrow{\eta_{GY}} & GFGY \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle G\epsilon_Y} \\ & & GY \end{array}$$

commute for every X in $\mathcal{C}$ and Y in $\mathcal{D}$.

Exercise 6.10. A functor $F\colon \mathcal{C} \to \mathbf{Set}$ is called *co-representable* if there exists an object X in $\mathcal{C}$ with $h^X \cong F$. We say that F is co-represented by X. Let $f_1, \dots, f_m \in \mathbf{Z}[X_1, \dots, X_n]$. Show that the functor

$$\mathbf{CRing} \to \mathbf{Set},\ R \mapsto \{x \in R^n \mid f_1(x) = \cdots = f_m(x) = 0\}$$

of Example 4.9 is co-representable.

Exercise 6.11 ($\star$)**.** Show that the functor

$$\mathrm{GL}_n\colon \mathbf{CRing} \to \mathbf{Set},\ R \mapsto \mathrm{GL}_n(R)$$

of Exercise 4.7 is co-representable. (Hint: first show that the functor $\mathrm{GL}_1\colon R \mapsto R^\times$ is isomorphic to h^{R_1} with $R_1 = \mathbf{Z}[X,Y]/(XY-1)$.) Let R_n be the commutative ring such that $\mathrm{GL}_n \cong h^{R_n}$. By the co-Yoneda lemma there is a unique ring homomorphism $R_1 \to R_n$ inducing the natural transformation $\det\colon \mathrm{GL}_n \to \mathrm{GL}_1$. Describe this ring homomorphism explicitly.