NOTES ON CATEGORY THEORY

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Date: 14 December, 2010.

Contents

0. Introduction

The aim of these notes is to provide an introduction to the language of Category Theory and a reference for the definitions of various "universal objects". The main content of the notes is contained in Sections 3 and 5 and indeed the first version of these notes consisted only of Section 3. The intent was simply to collect these definitions in one place in diagrammatic form. Soon after I added Section 5 to collect various concrete definitions. Section 1 and the introduction to Section 3 were added so that the notes would be self contained were I ever to share them. Finally Section 2 was added so that those who needed the material in Section 1 would be aware of a few more common terms from the language of Category Theory.

1. The Basics

There are three definitions in Category Theory that are of fundamental importance: Categories, Functors, and Natural Transformations. This section defines these and gives a few examples of each.

1.1. Categories.

Definition 1. A *category* $\mathcal{C}$ consists of three pieces of data that satisfy two additional conditions. The data is:

- A class of *objects* denoted $\mathrm{Ob}(\mathcal{C})$.
- For every $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class of *morphisms* or *arrows* denoted $\mathrm{Mor}_{\mathcal{C}}(X, Y)$.
- For every $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a binary operation called *composition* denoted $\circ\colon \mathrm{Mor}_{\mathcal{C}}(Y, Z) \times \mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{C}}(X, Z)$.

This data should satisfy:

- Composition is associative; that is, $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms f, g, and h such that the composition above is well defined.
- For every object X there is a distinguished morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ for all morphisms f and g such that the composition above is well defined.

There is some common notational sloppiness which would take undue effort to avoid so we mention it now and then take full advantage. While the notation $\mathrm{Ob}(\mathcal{C})$ is useful when we wish to be explicit it is more common to denote that X is an object of $\mathcal{C}$ by simply writing $X \in \mathcal{C}$. When we wish to say that f is a morphism in our category we will use the same notation $f \in \mathcal{C}$; it should be clear from context whether the item in question is an object or a morphism. Finally when the category is understood we will drop the subscript and simply write $\mathrm{Mor}(X, Y)$.

The notion of a category is highly abstract but as you can see from the following examples they are very familiar objects.

Example 1. The category of sets is denoted $\underline{\texttt{Set}}$. We let $\mathrm{Ob}(\underline{\texttt{Set}})$ be the class of all sets. For any two sets X and Y we take $\mathrm{Mor}(X, Y)$ to be the set of all maps from X to Y. Composition is given by standard composition of maps and $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ is the standard identity map.

Example 2. The category of groups is denoted $\underline{\texttt{Grp}}$. We let $\mathrm{Ob}(\underline{\texttt{Grp}})$ be the class of all groups. For any two groups G and H we take $\mathrm{Mor}(G, H)$ to be the set of all homomorphisms from G to H. Composition is given by standard composition of maps and $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ is the standard identity map.

Example 3. The category of topological spaces is denoted $\underline{\texttt{Top}}$. We let $\mathrm{Ob}(\underline{\texttt{Top}})$ be the class of all topological spaces. For any two spaces X and Y we take $\mathrm{Mor}(X, Y)$ to be the set of all continuous maps from X to Y. Composition is given by standard composition of maps and $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ is the standard identity map.

Example 4. The category of vector spaces over a field k is denoted $\underline{\texttt{Vect}}_k$. We let $\mathrm{Ob}(\underline{\texttt{Vect}}_k)$ be the class of all vector spaces over the field k. For any two vector spaces V and W we take $\mathrm{Mor}(V, W)$ to be the set of all k-linear maps from V to W. Composition is given by standard composition of maps and $\mathrm{id}_V \in \mathrm{Mor}(V, V)$ is the standard identity map. Similarly we can define $\underline{\texttt{fVect}}_k$ to be the category of finite dimensional vector spaces over a field k.

In each of the examples above the objects of the category are sets with (possibly) some additional structure and the morphisms are the set maps that preserve this structure. Informally categories of this type are called "concrete" (the precise definition of *concrete* follows Definition 7, we will use quotes till then). Most of the terminology and notation of category theory derives from "concrete" categories. A morphism $f \in \mathrm{Mor}(X, Y)$ has *domain* X and *codomain* Y, and we write $f: X \to Y$ or $X \xrightarrow{f} Y$ to indicate this. If the domain of f is equal to the codomain of g then we can form $f \circ g$ so f and g are *composable*. Finally the distinguished morphism $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ is called the *identity* morphism and is easily proven unique.

1.2. **Functors.** Now that we know what a category is we can talk about maps between categories, these are called functors.

Definition 2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* F from $\mathcal{C}$ to D assigns to every object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$ and to every morphism $f \in \mathrm{Mor}(X, Y)$ a morphism $F(f) \in \mathrm{Mor}(F(X), F(Y))$ such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathcal{C}$ and
- $F(f \circ g) = F(f) \circ F(g)$ for any composable morphisms $f, g \in \mathcal{C}$.

The most important fact about functors is that they take commutative diagrams in the category $\mathcal{C}$ to commutative diagrams in the category $\mathcal{D}$. As is the case with morphisms we say that $\mathcal{C}$ is the *domain*, $\mathcal{D}$ is the *codomain*, and write $F: \mathcal{C} \to \mathcal{D}$ or $\mathcal{C} \xrightarrow{F} \mathcal{D}$.

Functors come in two types, covariant and contravariant. What we have defined above is a covariant functor. In a contravariant functor the direction of the morphisms is reversed, so to $f \in \mathrm{Mor}(X, Y)$ we assign $F(f) \in \mathrm{Mor}(F(Y), F(X))$ and if f and g are composable then $F(f \circ g) = F(g) \circ F(f)$.

Often one thinks of morphisms in a category as the arrows in diagrams. A contravariant functor simply reverses the direction of each arrow it is applied to. It is standard to assume that functors are covariant unless otherwise specified and we will follow this convention. In addition to this, when a definition depends on a functor we will only state the covariant case. We leave it to the reader to "reverse the arrows" for the contravariant case.

Example 5. If $\mathcal{C}$ is any category then there is an identity functor $\mathrm{id}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ that assigns to each object that same object and to each map that same map.

Example 6. Define $F: \underline{\texttt{Vect}}_k \to \underline{\texttt{Set}}$ as follows: For any vector space V we let $F(V) \in \underline{\texttt{Set}}$ be the underlying set of elements and for any linear map $f: V \to W$

we let $F(f) \in \mathrm{Mor}(F(V), F(W))$ be f considered as a map of sets. This is an example of a forgetful functor. We will give a precise definition of forgetful functors in Section 2.3; for now simply note that in place of $\underline{\texttt{Vect}}_k$ we could easily have used $\underline{\texttt{Top}}$ or any other "concrete" category.

Example 7. Fix a field k and define $F\colon \underline{\texttt{Set}} \to \underline{\texttt{Vect}}_k$ as follows: For every set X we let $F(X)$ be the free vector space on X. Specifically the vectors in $F(X)$ are formal k-linear combinations $\sum_{x\in X} c_x x$ where each c_x is an element of k and for only finitely many x is $c_x \neq 0$. Vector addition is done by combining like terms and scalar multiplication by distributing over the formal sum. Given any set map $f\colon X \to Y$ we let $F(f)\colon F(X) \to F(Y)$ be the linear map induced by extending f k-linearly to all of $F(X)$; i.e.

$$F(f)\left(\sum_{x\in X} c_x x\right) = \sum_{x\in X} c_x f(x).$$

This is an example of a free functor (defined in Section 2.3).

Example 8. Fix a field k and define $-^*\colon \underline{\texttt{fVect}}_k \to \underline{\texttt{fVect}}_k$ as follows: For any k-vector space V we let $V^* = \mathrm{Hom}_k(V, k)$ be the dual of V. Given any linear map $f\colon V \to W$ define $f^*\colon W^* \to V^*$ by $f^*(T) = T \circ f$. This defines a contravariant functor from $\underline{\texttt{fVect}}_k$ to itself.

Given any two functors $F\colon \mathcal{D} \to \mathcal{E}$ and $G\colon \mathcal{C} \to \mathcal{D}$ we can define the functor $F \circ G\colon \mathcal{C} \to \mathcal{E}$ that assigns to every object $X \in \mathcal{C}$ the object $F(G(X)) \in \mathcal{E}$ and to every morphism $f \in \mathcal{C}$ the morphism $F(G(f)) \in \mathcal{E}$. This new functor is called the composition of the functors F and G.

Example 9. The composition of the functor $-^*$ from Example 8 with itself gives the functor $-^{**}\colon \underline{\texttt{fVect}}_k \to \underline{\texttt{fVect}}_k$ which sends every vector space V to its double dual V^{**}.

1.3. **Natural Transformations.** Continuing our descent into abstraction we consider maps between functors.

Definition 3. Let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$. A collection T of morphisms in $\mathcal{D}$, one morphism $T_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X))$ for each object $X \in \mathcal{C}$, is called a *natural transformation* if for any morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle T_X}\downarrow & & \downarrow{\scriptstyle T_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

commutes.

Again the notation $T\colon F \to G$ is standard. The morphism $T_X \in \mathcal{D}$ is called the component of T along X.

Example 10. Let $\mathrm{id}\colon \underline{\texttt{fVect}}_k \to \underline{\texttt{fVect}}_k$ be the identity functor from Example 5 and $-^{**}\colon \underline{\texttt{fVect}}_k \to \underline{\texttt{fVect}}_k$ the double dual from Example 9. Recall from linear algebra that for every vector v in a vector space V we can define a linear map

$\hat{v}\colon V^* \to k$ by $\hat{v}(f) = f(v)$. The maps $T_V\colon V \to V^{**}$ given by $v \mapsto \hat{v}$ are also linear and for any $f \in \mathrm{Mor}(V, W)$ the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle T_V}\downarrow & & \downarrow{\scriptstyle T_W} \\ V^{**} & \xrightarrow[f^{**}]{} & W^{**} \end{array}$$

commutes. Thus the collection T of these maps is a natural transformation from id to $-^{**}$.

2. Additional Definitions

The following are some additional terms that one may encounter. These definitions can be slightly different than what you are used to. For example if you ask someone who has studied algebra what an epimorphism is they will likely tell you it is a surjective homomorphism. In general a category need not be concrete so this definition is not "categorical". The appropriate generalization is found below.

2.1. **Types of Morphisms.**

Definition 4. Let $\mathcal{C}$ be a category and $f\colon X \to Y$ a morphism in $\mathcal{C}$.

- We say f is *epic*, or an *epimorphism*, if it is right-cancellative; that is, if $u \circ f = v \circ f$ implies $u = v$ for all $u, v \in \mathcal{C}$.
- We say f is *monic*, or a *monomorphism*, if it is left-cancellative; that is, if $f \circ u = f \circ v$ implies $u = v$ for all $u, v \in \mathcal{C}$.
- We say f is an isomorphism if there exists a morphism $f^{-1}\colon Y \to X$, called the inverse of f, such that $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$.
- The morphisms from an object to itself are called *endomorphisms*.
- The isomorphisms from an object to itself are called *automorphisms*.

The composition of two monomorphisms, epimorphisms, or isomorphisms is again a monomorphism, epimorphism, or isomorphism respectively. Two objects in a category are isomorphic if there is an isomorphism between them. Given an object $X \in \mathcal{C}$ the collection of endomorphisms of X is denoted $\mathrm{End}(X)$ and the collection of automorphisms of X is denoted $\mathrm{Aut}(X)$.

Any isomorphism is both epic and monic. In a concrete category any surjective morphism is epic and any injective morphism is monic. The converse of these three statements is true in some categories, for example `Set`, but in general this is not the case. We can find an easy counter example for two of the statements in `Top`. Consider the inclusion of a proper dense subspace into a topological space. Two continuous maps that agree on a dense subspace agree everywhere so this inclusion, while not surjective, is an epimorphism. Inclusions are injective so, while not an isomorphism, it is both epic and monic.

A non-trivial counter example showing that monics need not be injective is slightly harder to come by. For the reader interested in filling in the details themselves let `Div` be the full subcatagory of `Grp` consisting of divisible abelian groups (full subcategories are defined in the next section). The canonical factor homomorphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic in this category but not injective.

2.2. Subcategories.

Definition 5. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of two pieces of data:

- A subcollection $\mathrm{Ob}(\mathcal{D})$ of $\mathrm{Ob}(\mathcal{C})$ called the objects of the subcategory.
- For each $X, Y \in \mathrm{Ob}(\mathcal{D})$ a subcollection $\mathrm{Mor}_{\mathcal{D}}(X,Y)$ of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ called the morphisms of the subcategory.

This data should satisfy:

- For every $X \in \mathrm{Ob}(\mathcal{D})$ the identity $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ is contained in $\mathrm{Mor}_{\mathcal{D}}(X,X)$.
- The composition in $\mathcal{C}$ of two morphisms from $\mathcal{D}$ yields a morphism in $\mathcal{D}$.

The subcategory $\mathcal{D}$ is a category in its own right using the composition law from $\mathcal{C}$. If $\mathrm{Mor}_{\mathcal{C}}(X,Y) = \mathrm{Mor}_{\mathcal{D}}(X,Y)$ for every pair of objects $X, Y \in \mathcal{D}$ then we say that $\mathcal{D}$ is a full subcategory. The two conditions of a subcategory are trivial in this case so to specify a full subcategory we need only specify the subcollection of objects.

Example 11. From Example 4, $\underline{\texttt{fVect}}_k$ is a full subcategory of $\underline{\texttt{Vect}}_k$.

Example 12. The category of abelian groups is denoted $\underline{\texttt{Ab}}$. It is the full subcategory of $\underline{\texttt{Grp}}$ whose objects are the abelian groups in $\underline{\texttt{Grp}}$.

2.3. Natural Isomorphisms and Types of Functors.

Definition 6. Let $T\colon F \to G$ be a natural transformation. If each component morphism T_X is an isomorphism then T is called a *natural isomorphism* and F and G are *naturally isomorphic.*

Example 13. Let $T\colon \mathrm{id} \to -^{**}$ be the natural transformation from Example 10. It is a standard result of linear algebra that the components $T_V\colon V \to V^{**}$ are isomorphisms and therefore T is a natural isomorphism. This is the content of the word 'natural' when one says that a finite dimensional vector space is naturally isomorphic to its double dual.

Definition 7. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

- We say that F is an *isomorphism* if there is a functor $F^{-1}\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ F^{-1} = \mathrm{id}_{\mathcal{D}}$ and $F^{-1} \circ F = \mathrm{id}_{\mathcal{C}}$.
- We say that F is an *equivalence* if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{C}}$ respectively.
- We say that F is *full* if the mapping $f \mapsto F(f)$ induces a surjection $\mathrm{Mor}_{\mathcal{C}}(X,Y) \to \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$ for every pair of objects $X, Y \in \mathcal{C}$.
- We say that F is *faithful* if the mapping $f \mapsto F(f)$ induces an injection $\mathrm{Mor}_{\mathcal{C}}(X,Y) \to \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$ for every pair of objects $X, Y \in \mathcal{C}$.

A functor that is both full and faithful is called fully faithful. It can be shown that $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if F is fully faithful and for each $X \in \mathcal{D}$ there exists a $Y \in \mathcal{C}$ such that X is isomorphic to $F(Y)$. Any faithful functor of the form $F\colon \mathcal{C} \to \underline{\texttt{Set}}$ is called a *forgetful functor* and a category $\mathcal{C}$ is called *concrete* if it has such a functor. If C is an object of this concrete category then $F(C)$ is the *underlying set* of C.

Example 14. Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. There is a functor $F\colon \mathcal{D} \to \mathcal{C}$, called the inclusion functor, that takes each object/morphism of $\mathcal{D}$ to that same object/morphism considered as an element of $\mathcal{C}$. This functor is always faithful. It is full if and only if $\mathcal{D}$ is a full subcategory of $\mathcal{C}$.

Definition 8. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. The ordered pair of functors (F, G) is called an *adjoint pair* if there exist a collection of bijections $\tau_{C,D}\colon \mathrm{Mor}_{\mathcal{D}}(F(C), D) \to \mathrm{Mor}_{\mathcal{C}}(C, G(D))$, one for each $C \in \mathcal{C}$ and $D \in \mathcal{D}$, such that for any $f \in \mathrm{Mor}_{\mathcal{C}}(A, C)$ and $g \in \mathrm{Mor}_{\mathcal{D}}(D, B)$ the diagrams

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(F(C), D) & \xrightarrow{-\circ F(f)} & \mathrm{Mor}_{\mathcal{D}}(F(A), D) \\ \downarrow{\scriptstyle \tau_{C,D}} & & \downarrow{\scriptstyle \tau_{A,D}} \\ \mathrm{Mor}_{\mathcal{C}}(C, G(D)) & \xrightarrow[-\circ f]{} & \mathrm{Mor}_{\mathcal{D}}(A, G(D)) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(F(C), D) & \xrightarrow{g\circ -} & \mathrm{Mor}_{\mathcal{D}}(F(C), B) \\ \downarrow{\scriptstyle \tau_{C,D}} & & \downarrow{\scriptstyle \tau_{C,B}} \\ \mathrm{Mor}_{\mathcal{C}}(C, G(D)) & \xrightarrow[G(g)\circ -]{} & \mathrm{Mor}_{\mathcal{C}}(C, G(B)) \end{array}$$

commute.

If (F, G) is an adjoint pair then we say F is left-adjoint to G and G is right-adjoint to F. There are two natural transformations associated to every adjoint pair, the *unit* transformation, $\eta\colon \mathrm{id}_{\mathcal{C}} \to G \circ F$, whose component along $C \in \mathcal{C}$ is $\eta_C = \tau_{C,F(C)}(\mathrm{id}_{F(C)})$ and the *counit* transformation, $\varepsilon\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$, whose component along $D \in \mathcal{D}$ is $\varepsilon_D = \tau^{-1}_{G(D),D}(\mathrm{id}_{G(D)})$.

A given functor does not always have an adjoint but when it does that adjoint is essentially unique; that is, if

$$\tau'_{C,D}\colon \mathrm{Mor}_{\mathcal{D}}(F(C), D) \to \mathrm{Mor}_{\mathcal{C}}(C, G'(D))$$

is the bijection associated to another adjoint pair (F, G') then the transformation $R\colon G \to G'$ with component $R_D = \tau'_{G(D),D}(\varepsilon_D)$ along D is a natural isomorphism between G and G'. Similarly if (F', G) is another adjoint pair with unit transformation $\eta'\colon \mathrm{id}_{\mathcal{C}} \to G \circ F'$, then $L_C = \tau^{-1}_{C,F'(C)}(\eta'_C)$ gives a natural isomorphism $L\colon F \to F'$.

Adjoints can be composed. If (F, G) and (H, K) are two adjoint pairs with associated bijections

$$\tau_{C,D}\colon \mathrm{Mor}_{\mathcal{C}}(F(C), D) \to \mathrm{Mor}_{\mathcal{B}}(C, G(D)), \qquad \text{and}$$
$$\tau'_{D,E}\colon \mathrm{Mor}_{\mathcal{D}}(H(D), E) \to \mathrm{Mor}_{\mathcal{C}}(D, K(E))$$

then $\psi_{C,E} = \tau_{C,K(E)} \circ \tau'_{F(C),E}$ makes $(H \circ F, G \circ K)$ an adjoint pair.

A functor $F\colon \underline{\mathtt{Set}} \to \mathcal{C}$ is called *free* if it is left adjoint to a forgetful functor. An object of $\mathcal{C}$ is a *free object* or is free on X if it is of the form $F(X)$ for some set X.

Example 15. Let $G\colon \underline{\mathtt{Vect}}_k \to \underline{\mathtt{Set}}$ be the forgetful functor defined in Example 6 and $F\colon \underline{\mathtt{Set}} \to \underline{\mathtt{Vect}}_k$ the functor from Example 7. Maps out of vector spaces are defined by where they send the basis elements therefore restricting a linear map on

$F(X)$ to a set map on X is a bijection $\tau_{X,V}\colon \operatorname{Mor}(F(X), V) \to \operatorname{Mor}(X, G(V))$ and makes (F, G) an adjoint pair.

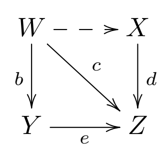
Example 16. Let $G\colon \underline{\texttt{Ab}} \to \underline{\texttt{Grp}}$ be the inclusion functor (see Example 14) and let $F\colon \underline{\texttt{Grp}} \to \underline{\texttt{Ab}}$ be abelianization; that is, $F(G) = G/[G, G]$ and for $f\colon G \to H$ the map $F(f)\colon G/[G, G] \to H/[H, H]$ is induced by factoring f to the quotient. If H is abelian then precomposition with the factor homomorphism $G \to G/[G, G]$ gives a bijection $\tau_{G,H}\colon \operatorname{Mor}(G/[G, G], H) \to \operatorname{Mor}(G, H)$ making (F, G) an adjoint pair.

3. Universal Properties

Universal properties have a general definition as initial and final properties in something called the comma category. This level of abstraction is not really necessary; for most purposes it suffices to know specific examples and not bother with the full generality. We take this approach here. This section is merely a collection of definitions for some common universal objects.

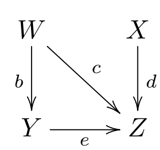
We make the following convention on diagrams. A diagram containing both solid and dotted edges is to be read initially as though the dotted edges are absent. When we claim the existence of morphisms *completing* the diagram, this means that if the dotted edges are labeled with these morphisms the entire diagram commutes. For example we could say

Given

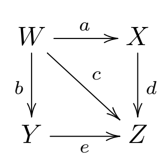


there exists a unique morphism $a\colon W \to X$ completing the diagram.

This means that if there are morphisms b, c, d, and e such that the diagram



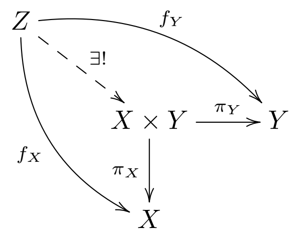
commutes then there exists a unique morphism a such that the diagram



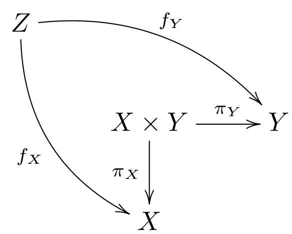
commutes.

Diagrams with dotted arrows can then be interpreted as if-then statements. Often some objects will be fixed and others arbitrary. We take the convention that any object or morphism not previously mentioned is quantified by a $\forall$ operator. For example the product of two objects in a category $\mathcal{C}$ is defined as follows:

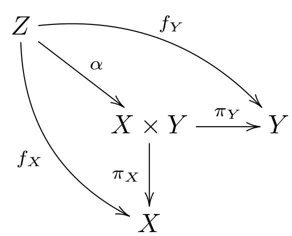
Let X and Y be objects in $\mathcal{C}$. Their product is an object $X \times Y$ together with morphisms $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ satisfying the universal property:



The object Z and morphisms f_X and f_Y were not mentioned before the diagram and so are quantified by $\forall$ operators. Thus the above states that for every object $Z \in \mathcal{C}$ with morphisms $f_X\colon Z \to X$ and $f_Y\colon Z \to Y$ such that the diagram



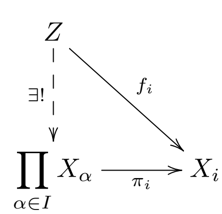
commutes there exists a unique morphism $\alpha\colon Z \to X \times Y$ such that the diagram



commutes. Note that the $\exists$ quantifiers of a diagram are asserted after the $\forall$ quantifier on the new elements of the diagram so the choice of morphism α depends on f_X and f_Y (and obviously on Z).

Universal objects are unique up to a canonical isomorphism so we will use "the" even though multiple objects in the category may satisfy the definition.

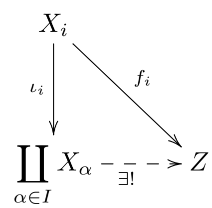
3.1. **Products.** Let $\mathcal{C}$ be a category and $\{X_\alpha\}_{\alpha \in I}$ a collection of objects in $\mathcal{C}$ indexed by some set I. The product of the X_α is an object $\prod_{\alpha \in I} X_\alpha$ together with morphisms $\left\{\pi_i\colon \prod_{\alpha \in I} X_\alpha \to X_i\right\}_{i \in I}$ satisfying the universal property:



So morphisms into the X_i factor through the product. Note that we've shown the diagram only for one X_i. We intend this diagram to represent the full diagram that shows all of the X_i; that is, the unique map $\alpha\colon Z \to \prod_{\alpha \in I} X_\alpha$ should satisfy $f_i = \pi_i \circ \alpha$ for all $i \in I$. If the set I is finite then we say that $\prod_{\alpha \in I} X_\alpha$ is a finite product.

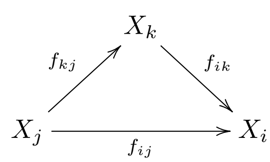
When I is finite, say $I = \{1, \ldots, n\}$, the notation $X_1 \times \cdots \times X_n$ is also used for the product, but never $\times_{\alpha \in I} X_\alpha$.

3.2. **Coproducts.** Let $\mathcal{C}$ be a category and $\{X_\alpha\}_{\alpha \in I}$ a collection of objects in $\mathcal{C}$ indexed by some set I. The coproduct of the X_α is an object $\coprod_{\alpha \in I} X_\alpha$ together with morphisms $\left\{\iota_i\colon X_i \to \coprod_{\alpha \in I} X_\alpha\right\}_{i \in I}$ satisfying the universal property:



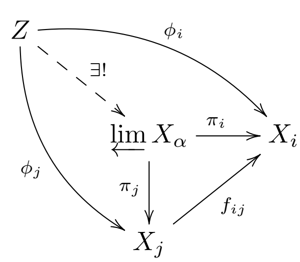
So morphisms out of the X_i factor through the coproduct. If the set I is finite then we say that $\coprod_{\alpha \in I} X_\alpha$ is a finite coproduct. Note that this is the exact same diagram from the definition of a product but with all the arrows reversed. For every universal object in category theory there is a co-object or categorical dual defined by taking the universal diagram and reversing all the arrows.

3.3. **Inverse Limits.** Let $\mathcal{C}$ be a category. A partially ordered set I is *directed* if there is an upper bound for every pair of elements; that is, for all $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. Let I be a directed set. Objects $\{X_i\}_{i \in I}$ together with morphisms $\{f_{ij}\colon X_j \to X_i\}_{i \leq j}$ are called an *inverse system* if the diagram



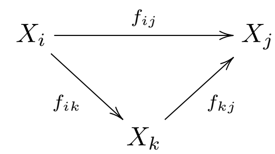
commutes for all $i \leq k \leq j$.

Let X be the inverse system with objects $\{X_i\}_{i \in I}$ and morphisms $\{f_{ij}\}_{i \leq j}$. The inverse limit, or projective limit, of this system is an object $\varprojlim X_\alpha$ together with morphisms $\left\{\pi_i\colon \varprojlim X_\alpha \to X_i\right\}$ satisfying the the universal property:



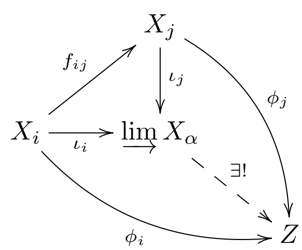
So morphisms into the inverse system factor through the inverse limit.

3.4. **Direct Limits.** Let $\mathcal{C}$ be a category and I a directed set. Objects $\{X_i\}_{i\in I}$ together with morphisms $\{f_{ij}\colon X_i \to X_j\}_{i\le j}$ are called a *direct system* if the diagram



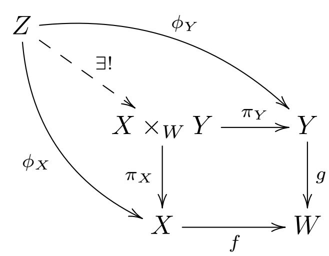
commutes for all $i \le k \le j$.

Let X be the direct system with objects $\{X_i\}_{i\in I}$ and morphisms $\{f_{ij}\}_{i\le j}$. The direct limit, or injective limit, of this system is an object $\varinjlim X_\alpha$ together with morphisms $\left\{\iota_i\colon X_i \to \varinjlim X_\alpha\right\}$ satisfying the the universal property:

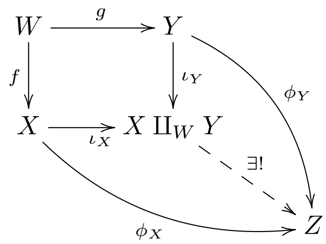


So morphisms out of the direct system factor through the direct limit.

3.5. **Pullbacks.** Let $\mathcal{C}$ be a category and $f\colon X \to W$ and $g\colon Y \to W$ morphisms. The pullback, or fibered product, of X and Y is an object $X \times_W Y$ together with morphisms $\pi_X\colon X \times_W Y \to X$ and $\pi_Y\colon X \times_W Y \to Y$ satisfying the universal property:



3.6. **Pushouts.** Let $\mathcal{C}$ be a category and $f\colon W \to X$ and $g\colon W \to Y$ morphisms. The pushout, or fibered coproduct, of X and Y is an object $X \amalg_W Y$ together with morphisms $\iota_X\colon X \to X \amalg_W Y$ and $\iota_Y\colon Y \to X \amalg_W Y$ satisfying the universal property:



3.7. **Final Objects.** Let $\mathcal{C}$ be a category. The final object is an object F satisfying the universal property:

$$Z \overset{\exists!}{\dashrightarrow} F$$

So every object has exactly one morphism into F.

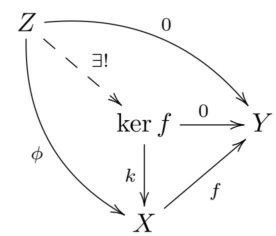
3.8. **Initial Objects.** Let $\mathcal{C}$ be a category. The initial object is an object I satisfying the universal property:

$$I \overset{\exists!}{\dashrightarrow} Z$$

So there is exactly one morphism from I into any object.

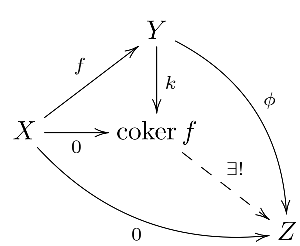
3.9. **Zero Objects and Morphisms.** Let $\mathcal{C}$ be a category. The zero or null object is an object $0_{\mathcal{C}}$ that is both initial and final. The unique map into or out of a zero object is called the zero morphism and is also denoted 0. Given any two objects X and Y the composition $X \to 0_{\mathcal{C}} \to Y$ is called the zero map from X to Y and is again denoted 0.

3.10. **Kernels.** Let $\mathcal{C}$ be a category with a zero object and morphism $f\colon X \to Y$. The kernel of f is an object $\ker f$ together with morphism $k\colon \ker f \to X$ satisfying $f \circ k = 0$ and the universal property:



Immediately one has that k is monic. If f is monic then $\ker f$ is the final object in $\mathcal{C}$. The kernel of a zero morphism is the identity morphism of the domain. Here, as elsewhere, term 'kernel' can mean either the object $\ker f$, the morphism $k\colon \ker f \to X$, or the pair of the two. When only the object is given the morphism should be clear from context.

3.11. **Cokernels.** Let $\mathcal{C}$ be a category with a zero object and morphism $f\colon X \to Y$. The cokernel of f is an object $\operatorname{coker} f$ together with morphism $k\colon Y \to \operatorname{coker} f$ satisfying $k \circ f = 0$ and the universal property:



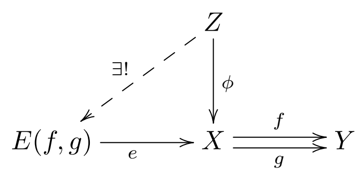
Immediately one has that k is epic. If f is epic then $\operatorname{coker} f$ is the initial object in $\mathcal{C}$. The cokernel of a zero morphism is the identity morphism of the codomain. As with kernels the term 'cokernel' can mean either the object, the morphism, or the pair of the two.

3.12. **Biproducts.** Let $\mathcal{C}$ be a category with zero morphisms (as in Section 3.9 or Section 4.1) and let $\{X_\alpha\}_{\alpha\in I}$ be a finite collection of objects. The biproduct of the X_α is an object $\bigoplus_{\alpha\in I} X_\alpha$ together with morphisms $\left\{\pi_i\colon \bigoplus_{\alpha\in I} X_\alpha \to X_i\right\}_{i\in I}$ and $\left\{\iota_i\colon X_i \to \bigoplus_{\alpha\in I} X_\alpha\right\}_{i\in I}$ such that the following properties hold:

- $\bigoplus_{\alpha\in I} X_\alpha$ together with $\{\pi_i\}_{i\in I}$ is a product.
- $\bigoplus_{\alpha\in I} X_\alpha$ together with $\{\iota_i\}_{i\in I}$ is a coproduct.
- The composition $\pi_i \circ \iota_j$ equals id_{X_i} if $i = j$ and 0 otherwise.

Immediately one has that the π_i are epic and the ι_i are monic. Note that nothing in the definition explicitly requires the collection I to be finite, but in practice only finite biproducts exist or are ever assumed to exist so here we just define them as such.

3.13. **Equalizers.** Let $\mathcal{C}$ be a category and $f, g\colon X \to Y$ two morphisms. The equalizer (also called a difference kernel) of f and g is an object $E(f, g)$ together with a morphism $e\colon E(f, g) \to X$ satisfying the following universal property:



Immediately one has that e is monic.

4. Enriched Categories

Informally a category $\mathcal{C}$ is enriched in $\mathcal{D}$ if for any two objects $X, Y \in \mathcal{C}$ the morphism class $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is an object of the category $\mathcal{D}$ in such a way that the two structures interact nicely. Full generality here is unnecessary; for most purposes it suffices to know the examples below in which categories are enriched in Ab.

4.1. **Ab-categories.**

Definition 9. An *Ab-category* is a category $\mathcal{C}$ such that:

- Each $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is an abelian group.
- Composition of morphisms is bilinear.

We write the group operation on each morphism set as addition. The condition that composition of morphisms is bilinear then means that for all morphisms $f_1, f_2 \in \mathrm{Mor}(X, Y)$, $g \in \mathrm{Mor}(Y, Z)$, and $h \in \mathrm{Mor}(W, X)$ the identities

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2 \quad \text{and} \quad (f_1 + f_2) \circ h = f_1 \circ h + f_2 \circ h$$

hold.

Example 17. Both Ab and $\mathtt{Vect}_k$ are Ab-categories. The abelian group structure on each $\mathrm{Mor}(X, Y)$ is given by pointwise addition.

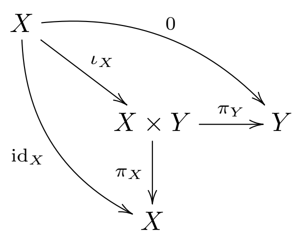
Two morphisms f and g in $\mathcal{C}$ are called parallel if they have the same domain and codomain. The identity object of $\mathrm{Mor}(X, Y)$ is called the zero morphism from X to Y and is denoted 0, or $0_{(X,Y)}$ if the domain and range need be notated. For any morphisms $f\colon W \to X$ and $g\colon Y \to Z$ one can show $0_{(X,Y)} \circ f = 0_{(W,Y)}$ and $g \circ 0_{(X,Y)} = 0_{(X,Z)}$. When $\mathcal{C}$ has a zero object the zero morphism $0_{(X,Y)}$ just defined is exactly the zero morphism $X \to 0 \to Y$ defined in Section 3.9.

For every object $X \in \mathcal{C}$ the abelian group structure on $\mathrm{End}(X) = \mathrm{Mor}(X, X)$ becomes a ring structure with multiplication given by composition of morphisms. This is called the endomorphism ring of X or associated to X. If two objects are isomorphic then so are their associated rings. The ring isomorphism is given by $f \circ - \circ f^{-1} \colon \mathrm{End}(X) \to \mathrm{End}(Y)$ where $f \colon X \to Y$ is any isomorphism in $\mathcal{C}$.

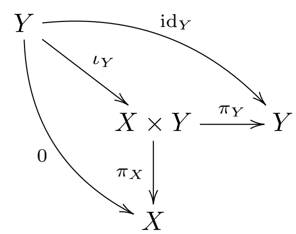
Definition 10. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is called *additive* if $F(f + g) = F(f) + F(g)$ whenever f and g are parallel.

Example 18. Every object $A \in \mathcal{C}$ induces an additive functor $\mathrm{Mor}(A, -) \colon \mathcal{C} \to \underline{\mathtt{Ab}}$ which maps an object X to the abelian group $\mathrm{Mor}(A, X)$ and a morphism $f \colon X \to Y$ to the homomorphism $\mathrm{Mor}(A, f) \colon \mathrm{Mor}(A, X) \to \mathrm{Mor}(A, Y)$, more commonly denoted $f \circ -$, given by post-composition with f (i.e. $\phi \mapsto f \circ \phi$). Similarly, A induces an additive contravariant functor $\mathrm{Mor}(-, A)$, using pre-composition $- \circ f \colon \mathrm{Mor}(Y, A) \to \mathrm{Mor}(X, A)$ instead of post.

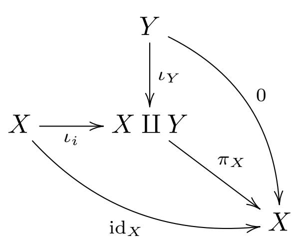
Kernels and equalizers are identical; that is, if either of $E(f, g)$ or $\ker(f - g)$ exists then they both exist and are canonically isomorphic. This is also true of products, coproducts, and biproducts. If the product $X \times Y$ exists then its universal property gives unique maps ι_X and ι_Y making the diagrams



and



commute. These maps make $X \times Y$ into a coproduct. Similarly if the coproduct $X \amalg Y$ exists then its universal property gives unique maps π_X and π_Y making the diagrams



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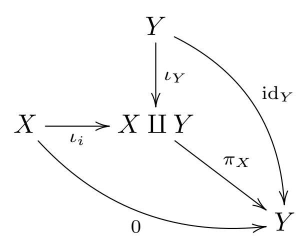

commute. These maps make $X \amalg Y$ into a product. In either case the resulting morphisms satisfy the additional condition required of a biproduct.

4.2. **Additive Categories.**

Definition 11. An *additive* category is a category $\mathcal{C}$ such that:

- Each $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is an abelian group.
- Composition of morphisms is bilinear.
- There is a zero object in $\mathcal{C}$.
- Any two objects have a biproduct.

So an additive category is simply an Ab-category with a zero and biproducts. Note that when checking that a particular category satisfies these axioms it suffices to check that every pair of objects has either a product or a coproduct as these will automatically be biproducts.

Example 19. The trivial group is the zero object of $\underline{\texttt{Ab}}$ and the direct sum is a biproduct so $\underline{\texttt{Ab}}$, and similarly $\underline{\texttt{Vect}}_k$, are additive categories.

An additive functor between two additive categories takes the zero object in the domain to the zero object in the codomain. It also takes any zero morphism to a zero morphism.

Let $f\colon \bigoplus_{i=1}^m X_i \to \bigoplus_{j=1}^n Y_j$ be a morphism between two biproducts and let $\left\{\iota_i\colon X_i \to \bigoplus_{j=1}^m X_j\right\}_{i=1}^m$ and $\left\{\pi_j\colon \bigoplus_{i=1}^n Y_i \to Y_j\right\}_{j=1}^n$ be the inclusions and projections respectively. Then $f_M = [\pi_i \circ f \circ \iota_j]$, called the matrix representation of f, is an $n \times m$ matrix whose $(i,j)^{\text{th}}$ entry is a morphism in $\text{Mor}(X_j, Y_i)$. Conversely given any such a matrix $[f_{ij}]$ the universal properties of biproducts ensure that there exists a unique morphism $f\colon \bigoplus_{i=1}^m X_i \to \bigoplus_{j=1}^n Y_j$ such that $f_M = [f_{ij}]$. If f and g are either parallel or composable morphisms then the sum or product respectively of the matrixes f_M and g_M (using the standard matrix formulas) will be well defined and satisfy $f_M + g_M = (f+g)_M$ and $f_M \cdot g_M = (f \circ g)_M$.

4.3. **Abelian Categories.**

Definition 12. An *abelian* category is a category $\mathcal{C}$ such that:

- Each $\text{Mor}_{\mathcal{C}}(X,Y)$ is an abelian group.
- Composition of morphisms is bilinear.
- There is a zero object in $\mathcal{C}$.
- Any two objects have a biproduct.
- Every morphism has a kernel and a cokernel.
- Every monic is a kernel and every epic is a cokernel.

So an abelian category is simply an additive category in which all kernels and cokernels exist, all monics are kernels, and all epics are cokernels. When every morphism has a kernel and a cokernel the last condition is equivalent to the statement that for every monic f and for every epic g we have $\ker(\operatorname{coker} f) = f$ and $\operatorname{coker}(\ker g) = g$. Some authors define abelian categories using this condition instead.

Example 20. In $\underline{\texttt{Ab}}$ any morphism $f\colon X \to Y$ has kernel $f^{-1}(0)$ and cokernel $Y/\operatorname{im} f$. The monic and epic morphisms are exactly the injective and surjective homomorphisms respectively and one can easily check the alternate final condition on these maps. Thus $\underline{\texttt{Ab}}$, and similarly $\underline{\texttt{Vect}}_k$, are abelian categories.

In an abelian category pullbacks and pushouts always exist. A morphism is monic if and only if its kernel is zero, epic if and only if its cokernel is zero, and an isomorphism if and only if it is both monic and epic.

5. Common Categories

The following is a list of some common categories along with the constructions for the universal objects that they poses.

5.1. **Sets.**

- Denoted: Set
- Objects: All sets.
- $\mathrm{Mor}(X, Y)$: Set maps $f : X \to Y$.
- Product: Cartesian product.
- Coproduct: Disjoint union.
- Inverse Limit: Subset $\{(a_i) \mid a_i = f_{ij}(a_j)\ \forall i \le j\}$ of the product.
- Direct Limit: Coproduct modulo $x_i \sim x_j$ if $f_{ik}(x_i) = f_{jk}(x_j)$ for some k.
- Pullback: Subset $\{(x, y) \mid f(x) = g(y)\}$ of the product.
- Pushout: Disjoint union modulo $f(w) \sim g(w)$ for all $w \in W$.
- Initial object: The empty set.
- Final object: Any singleton.
- Zero object: None.
- Equalizer: Subset $\{x \mid f(x) = g(x)\}$ of the domain.

5.2. **Groups.**

- Denoted: Grp
- Objects: All groups.
- $\mathrm{Mor}(G, H)$: Group homomorphisms $f : G \to H$.
- Product: Direct product.
- Coproduct: Free product.
- Inverse Limit: Subgroup $\{(a_i) \mid a_i = f_{ij}(a_j)\ \forall i \le j\}$ of the product.
- Direct Limit: Coproduct modulo $x_i \sim x_j$ if $f_{ik}(x_i) = f_{jk}(x_j)$ for some k.
- Pullback: Subgroup $\{(x, y) \mid f(x) = g(y)\}$ of the product.
- Pushout: Free product with amalgamation.
- Zero object: Trivial group.
- Kernel: Subgroup $f^{-1}(\text{identity})$.
- Cokernel: Factor group $Y/\overline{\operatorname{im} f}$ where $\overline{\operatorname{im} f}$ is the normal closure of $\operatorname{im} f$.
- Equalizer: Subgroup $\{x \mid f(x) = g(x)\}$ of the domain.

5.3. **Abelian Groups.**

- Denoted: Ab
- Objects: Abelian groups.
- $\mathrm{Mor}(G, H)$: Group homomorphisms $f : G \to H$.
- Product: Direct product.
- Coproduct: Direct sum.
- Inverse Limit: Subgroup $\{(a_i) \mid a_i = f_{ij}(a_j)\ \forall i \le j\}$ of the product.
- Direct Limit: Coproduct modulo $x_i \sim x_j$ if $f_{ik}(x_i) = f_{jk}(x_j)$ for some k.
- Pullback: Subgroup $\{(x, y) \mid f(x) = g(y)\}$ of the product.
- Pushout: Direct sum modulo $\langle (f(w), -g(w)) \mid w \in W \rangle$.
- Zero object: Trivial group.
- Kernel: Subgroup $f^{-1}(\text{identity})$.
- Cokernel: Factor group $Y/\operatorname{im} f$.
- Biproduct: Exists.
- Equalizer: Kernel of $f - g$.

5.4. **Rings.**

- Denoted: Rng
- Objects: All rings (with unity).

- Mor(R, S): Ring homomorphisms $f\colon R \to S$ (respecting unity).
- Product: Direct product.
- Coproduct: A construction analogous to the free product of groups.
- Inverse Limit: Subring $\{(a_i) \mid a_i = f_{ij}(a_j)\ \forall i \leq j\}$ of the product.
- Direct Limit: Coproduct modulo $x_i \sim x_j$ if $f_{ik}(x_i) = f_{jk}(x_j)$ for some k.
- Pullback: Subring $\{(x, y) \mid f(x) = g(y)\}$ of the product.
- Pushout: Coproduct modulo $f(w) \sim g(w)$.
- Initial object: Ring of integers $\mathbb{Z}$.
- Final object: Trivial ring $(0 = 1)$.
- Zero object: None.
- Equalizer: Subring $\{x \mid f(x) = g(x)\}$ of the domain.

5.5. **Commutative Rings.**

- Denoted: <u>`CRng`</u>
- Objects: Commutative rings (with unity).
- Mor(R, S): Ring homomorphisms $f\colon R \to S$ (respecting unity).
- Product: Direct product.
- Coproduct: Tensor Product.
- Inverse Limit: Subring $\{(a_i) \mid a_i = f_{ij}(a_j)\ \forall i \leq j\}$ of the product.
- Direct Limit: Coproduct modulo $x_i \sim x_j$ if $f_{ik}(x_i) = f_{jk}(x_j)$ for some k.
- Pullback: Subring $\{(x, y) \mid f(x) = g(y)\}$ of the product.
- Pushout: Tensor product modulo $f(w) \sim g(w)$.
- Initial object: Ring of integers $\mathbb{Z}$.
- Final object: Trivial ring $(0 = 1)$.
- Zero object: None.
- Equalizer: Subring $\{x \mid f(x) = g(x)\}$ of the domain.

5.6. **Modules.** Fix a ring A.

- Denoted: $A -$ <u>`Mod`</u>
- Objects: All left A-modules.
- Mor(M, N): A-module homomorphisms $f\colon X \to Y$.
- Product: Direct product.
- Coproduct: Direct sum.
- Inverse Limit: Submodule $\{(a_i) \mid a_i = f_{ij}(a_j)\ \forall i \leq j\}$ of the product.
- Direct Limit: Coproduct modulo $x_i \sim x_j$ if $f_{ik}(x_i) = f_{jk}(x_j)$ for some k.
- Pullback: Submodule $\{(x, y) \mid f(x) = g(y)\}$ of the product.
- Pushout: Direct sum modulo $\langle (f(w), -g(w)) \mid w \in W \rangle$.
- Zero object: Trivial module.
- Kernel: $\ker f = f^{-1}(0)$.
- Cokernel: Quotient module $Y / \operatorname{im} f$.
- Biproduct: Exists.
- Equalizer: Kernel of $f - g$.

5.7. **Topological Spaces.**

- Denoted: <u>`Top`</u>
- Objects: All topological spaces.
- Mor(X, Y): Continuous maps $f\colon X \to Y$.
- Product: Product of the underlying sets with the product topology.
- Coproduct: Disjoint union of the X_α with the disjoint union topology.

- Inverse Limit: Inverse limit in Set with the initial topology.
- Direct Limit: Direct limit in Set with the final topology.
- Pullback: Subspace $\{(x, y) \mid f(x) = g(y)\}$ of the product.
- Pushout: Disjoint union modulo $f(w) \sim g(w)$.
- Initial object: The empty space.
- Final object: Any singleton.
- Zero object: None.
- Equalizer: Subspace $\{x \mid f(x) = g(x)\}$ of the domain.

5.8. **Pointed Topological Spaces.**

- Denoted: $\texttt{Top}_*$
- Objects: Pairs (X, x) with $X \in$ Top and $x \in X$ (called the base point).
- $\mathrm{Mor}((X, x), (Y, y))$: Continuous maps $f\colon X \to Y$ such that $f(x) = y$.
- Product: Product in Top based at the product of the base points.
- Coproduct: Wedge sum with the base points identified.
- Inverse Limit: Inverse limit in Set with the initial topology.
- Direct Limit: Direct limit in Set with the final topology.
- Pullback: Subspace $\{(x, y) \mid f(x) = g(y)\}$ of the product.
- Pushout: Disjoint union modulo $f(w) \sim g(w)$.
- Zero object: Any singleton.
- Kernel: Subspace f^{-1}(base point) based at the base point of X.
- Cokernel: Quotient space $Y/\overline{\mathrm{im}\, f}$ based at $\overline{\mathrm{im}\, f}$.
- Equalizer: Subspace $\{x \mid f(x) = g(x)\}$ of the domain.

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