Work in Evidence of Talent

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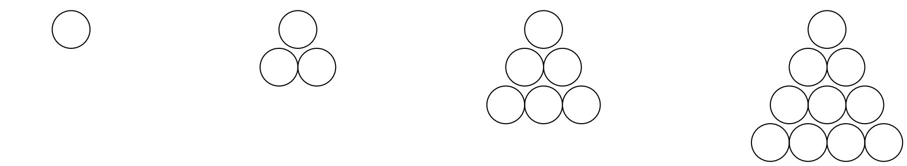
Introduction

This is a summary of the research I have conducted to fulfill the "Work in Evidence of Talent" portion of my application to the UCSB College of Creative Studies mathematics program. It involves experimentation with n^{th} dimensional trigonal sequences, curious series expansions, and series notations of simple exponents, and includes an applied math problem which requires use of geometry and advanced calculus knowledge. All included work is my own. I am unsure whether any of the topics covered are open questions, but the goal of this installment is to demonstrate my capabilities rather than to advance mathematical knowledge.

1 Trigonal Sequences

One of the most well-known sequences are the “triangular” numbers. This sequence is derived by starting with a single object, and adding rows of increasing numbers of such objects in the form of an equilateral triangle. It is easy to explain this graphically:



$T_2(1) = 1$ $T_2(2) = 3$ $T_2(3) = 6$ $T_2(4) = 10$

$T_2(x)$ can be found by summing the first x natural numbers: $1 + 2 + 3 + ... + x$. This can be denoted more elegantly as:

$$T_2(x)\text{[1]} = \sum_{n=1}^{x} n$$

The next echelon of this sort of sequence is the “tetrahedral” numbers. These are similar to the triangular numbers but are formed through iterations of expanding tetrahedra rather than triangles. In essence, the tetrahedral numbers can be imagined as increasing layers of triangular numbers, stacked on top of each other. Each progressive layer of such a tetrahedron is a sequential triangular number. As such, the general form for the tetrahedral numbers is:

$$T_3(x) = \sum_{m=1}^{x} T_2(m) = \sum_{m=1}^{x} \sum_{n=1}^{m} n$$

[1] Note that the subscript, called δ, of $T_2(x)$ and $T_3(x)$ refers to the dimension of the shape or solid used to form these numbers graphically (eg. $T_2(x)$ refers to the triangular sequence because $\delta = 2$ and triangles are 2-dimensional). $T_1(x)$ then indicates the sequence derived from progressively expanding a one-dimensional object – a line. So, $T_1(x) = x$. By the same logic, $T_0(x)$ is non-dimensional, so it is formed by a single point. Therefore, $T_0(x) = 1$ for all x.

Triangles are so closely-related to summations because of the simple property that their width grows steadily as their height increases. So, the existence of further dimensions of trigonal sequences can be extrapolated, each having the form:

$$T_\delta(x) = \sum_{n=1}^{x} T_{\delta-1}(n),$$

where δ denotes the referenced trigonal object's dimensions. Each sequential δ value implies that the series indicated is arithmetically formed through $\delta - 1$ summations:

$$\begin{aligned}
T_1(x) &= x; \\
T_2(x) &= \sum_{n=1}^{x} n \\
T_3(x) &= \sum_{m=1}^{x} \sum_{n=1}^{m} n \\
T_4(x) &= \sum_{\ell=1}^{x} \sum_{m=1}^{\ell} \sum_{n=1}^{m} n \\
&\textit{...etc.}
\end{aligned}$$

It is possible to evaluate each of these as a polynomial through use of the elementary formulas for sums of consecutive powers. For instance, it is clear that

$$T_2(x) = \sum_{n=1}^{x} n = \frac{x(x+1)}{2}.$$

Furthermore, by plugging in the polynomial form of $T_2(x)$,

$$\begin{aligned} T_3(x) &= \sum_{m=1}^{x} \frac{m^2+m}{2} \\ &= \frac{1}{2}\sum_{m=1}^{x} m^2 + \frac{1}{2}\sum_{m=1}^{x} m \\ &= \frac{x(x+1)(2x+1)}{12} + \frac{x(x+1)}{4} \\ &= \frac{x(x+1)(x+2)}{6}. \end{aligned}$$

By the same reasoning,

$$\begin{aligned} T_4(x) &= \sum_{\ell=1}^{x} \frac{\ell(\ell+1)(\ell+2)}{6} \\ &= \left(\frac{1}{6}\right)\left(\frac{x^2(x+1)^2}{4}\right) + \left(\frac{3}{6}\right)\left(\frac{x(x+1)(2x+1)}{6}\right) + \left(\frac{2}{6}\right)\left(\frac{x(x+1)}{2}\right) \\ &= \frac{x(x+1)(x+2)(x+3)}{24} \\ T_5(x) &= \sum_{k=1}^{x} \frac{k(k+1)(k+2)(k+3)}{24} \\ &= \left(\frac{1}{24}\right)\left(\frac{x(x+1)(2x+1)(3x^2+3x-1)}{30}\right) + \left(\frac{6}{24}\right)\left(\frac{x^2(x+1)^2}{4}\right) \\ &+ \left(\frac{11}{24}\right)\left(\frac{x(x+1)(2x+1)}{6}\right) + \left(\frac{6}{24}\right)\left(\frac{x(x+1)}{2}\right) \\ &= \frac{x(x+1)(x+2)(x+3)(x+4)}{120} \end{aligned}$$

Omitting the lengthy algebra as it becomes incredibly space-consuming,

$$T_6(x) = \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{720}$$
$$T_7(x) = \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)(x+6)}{5040}$$
$$T_8(x) = \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)(x+6)(x+7)}{40320}$$

The trend is observed to be:

$$T_\delta(x) = \frac{\prod_{n=1}^{\delta}(x+n-1)}{\delta!}$$

The following table contains values of certain trigonal sequences:

x	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10	11	12	13	
3	1	3	6	10	15	21	28	36	45	55	66	78		
4	1	4	10	20	35	56	84	120	165	220	286			
5	1	5	15	35	70	126	210	330	495	715				
6	1	6	21	56	126	252	462	792	1287					
7	1	7	28	84	210	462	924	1716						
8	1	8	36	120	330	792	1716							
9	1	9	45	165	495	1287								
10	1	10	55	220	715									
11	1	11	66	286										
12	1	12	78											
13	1	13												
14	1													

Note that in the above orientation, these sequences form Pascal's triangle. The triangle is rotated counterclockwise somewhat, such that each of the triangle's rows is a north-eastern diagonal of the table. This makes sense, as

$$T_\delta(x) = \sum_{n=1}^{x} T_{\delta-1}(n) = T_{\delta-1}(x) + \sum_{n=1}^{x-1} T_{\delta-1}(n) = T_{\delta-1}(x) + T_\delta(x-1),$$
$$T_0(x) = 1, for\ all\ x,$$
$$T_\delta(1) = 1, for\ all\ \delta$$

So, each border is composed only of 1s, and each cell in the above table is the sum of the cell above and the cell to the left of it. This is identical to the formation algorithm of Pascal's triangle. As such, the function for evaluating each element of the triangle applies:

$$P(n,k) = \binom{n}{k} = \frac{n!}{r!(n-k)!}$$

where n is the row of Pascal's triangle that contains the element and k is the element's number. In the context of the tabular representation, $x+\delta-1$ coincides with n, and δ coincides with k. So,

$$\begin{aligned}
T_\delta(x) &= \binom{x+\delta-1}{\delta} \\
&= \frac{(x+\delta-1)!}{\delta!(x+\delta-1-\delta)!} \\
&= \frac{(x+\delta-1)!}{(x-1)!} * \frac{1}{\delta!} \\
&= [(x)*(x+1)*(x+2)***(x+\delta-1)] * \frac{1}{\delta!} \\
&= \prod_{n=1}^{\delta}(x+n-1) * \frac{1}{\delta!} \\
T_\delta(x) &= \frac{\prod_{n=1}^{\delta}(x+n-1)}{\delta!}
\end{aligned}$$

This proves the original statement.

This has applications in modulo arithmetic. Because of the closed property of integer addition, any $T_\delta(x)$ is integral. So, $d!$ divides $\prod_{n=1}^{\delta}(x+n-1)$ for any δ and any x. That is,

$$\prod_{n=1}^{\delta}(x+n-1) \equiv 0 \ (mod\ \delta!)$$

Another application of these sequences is found through upward expansion of Pascal's triangle, as seen below. By following the same algorithm for creating the triangle, but reversed, one can discover another sign-alternating, flipped triangle just above the original. The key thing to notice here is that the triangle's orientation is flipped, such that the diagonals of the original triangle are now rows.

$$\left(\begin{array}{ccccccccccccccccccc}
-7 & & . & & . & & . & & . & & . & & 1 & & & & & & \\
-6 & . & & . & & . & & . & & . & & 1 & & -6 & & & & & \\
-5 & & . & & . & & . & & . & & 1 & & -5 & & 15 & & & & \\
-4 & . & & . & & . & & . & & 1 & & -4 & & 10 & & -20 & & & \\
-3 & & . & & . & & . & & 1 & & -3 & & 6 & & -10 & & 15 & & \\
-2 & . & & . & & . & & 1 & & -2 & & 3 & & -4 & & 5 & & -6 & \\
-1 & & . & & . & & 1 & & -1 & & 1 & & -1 & & 1 & & -1 & & 1 \\
0 & . & & . & & 1 & & . & & . & & . & & . & & . & & . & \\
1 & & . & & 1 & & 1 & & . & & . & & . & & . & & . & & . \\
2 & . & & 1 & & 2 & & 1 & & . & & . & & . & & . & & . & \\
3 & & 1 & & 3 & & 3 & & 1 & & . & & . & & . & & . & & .
\end{array}\right)$$

Now, it is known that each diagonal of the original Pascal's triangle is a trigonal sequence. So, one can extrapolate that, by placing an infinite number of zeroes, followed by an alternating trigonal sequence starting with a positive 1, one can easily originate negative rows of Pascal's triangle.

2 A Curio

Consider polynomial division of the equal functions $\frac{1}{1-x}$ and $\frac{-1}{x-1}$.

$$\frac{1}{1-x} = \frac{-1}{x-1}$$

By polynomial division,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$
$$and$$
$$\frac{-1}{x-1} = \frac{-1}{x} + \frac{-1}{x^2} + \frac{-1}{x^3} + ...$$

Because the two are equal, we have:

$$[1 + x + x^2 + x^3 + ...] = [\frac{-1}{x} + \frac{-1}{x^2} + \frac{-1}{x^3} + ...]$$

By adding each term on the right-hand side of this equation, we are left with:

$$[1 + x + x^2 + x^3 + ...] + [\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + ...] = 0$$

Note the negation of each fractional term. This equates to:

$$[...\frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + x^3 + ...] = 0$$
$$or$$
$$\sum_{n=-\infty}^{\infty} x^n = 0$$

A shocking assertion, as each term of the summation is positive or approaches zero, so its cancelation to zero makes little sense.

Just to make sure there is no mistake in the polynomial division, it would be be a good idea to check this by finding each function's Maclaurin series, calling the former 'f' and the latter 'g':

$$\begin{aligned} f &= (1-x)^{-1} & f(0) &= 1 \\ f^{i} &= (1-x)^{-2} & f^{i}(0) &= 1 \\ f^{ii} &= 2(1-x)^{-3} & f^{ii}(0) &= 2 \\ f^{iii} &= 6(1-x)^{-4} & f^{iii}(0) &= 6 \\ f^{iv} &= 24(1-x)^{-5} & f^{iv}(0) &= 24 \\ f^{v} &= 120(1-x)^{-6} & f^{v}(0) &= 120 \end{aligned}$$

It is evident that the value of each $f^{(n)}(0)$ is $n!$. So, the Maclaurin series is derived as follows:

$$\begin{aligned} f &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0) * x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{n! * x^n}{n!} \\ &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + ... \end{aligned}$$

So, the division was done correctly for f. Moving on to g:

$$\begin{aligned} g &= -(x-1)^{-1} & g(0) &= 1 \\ g^{i} &= (x-1)^{-2} & g^{i}(0) &= 1 \\ g^{ii} &= -2(x-1)^{-3} & g^{ii}(0) &= 2 \\ g^{iii} &= 6(x-1)^{-4} & g^{iii}(0) &= 6 \\ g^{iv} &= -24(x-1)^{-5} & g^{iv}(0) &= 24 \\ g^{v} &= 120(x-1)^{-6} & g^{v}(0) &= 120 \end{aligned}$$

Once again, each $g^{(n)}(0) = n!$ Therefore, g yields the same Maclaurin series as f does. While this may not have confirmed the polynomial division of g, it at least makes us a little more certain that $f = g$. Currently, I know of no easy way to reconfirm the division, as g's quotient includes negative exponents of x, which is impossible to recreate through Taylor series.

Disclaimer:

$$\sum_{n=-\infty}^{\infty} x^n \neq 0$$

Unfortunately, the contention that this summation equals zero is untrue. The issue is easily identifiable in the statement's derivation:

$$\begin{aligned}
\frac{1}{1-x} &= \frac{-1}{x-1} \\
[1 + x + x^2 + x^3 + ...] &= [\frac{-1}{x} + \frac{-1}{x^2} + \frac{-1}{x^3} + ...] \\
\sum_{n=0}^{\infty} x^n &= -\sum_{n=1}^{\infty} \left(\frac{1}{x}\right)^n
\end{aligned}$$

These two resultant series are each geometric series with ratios x and $\frac{1}{x}$ respectively. However, in order for a geometric series to be convergent, and therefore algebraically workable, the absolute value of its ratio must be strictly less than 1. In context:

$$|x| < 1 \qquad \textit{and} \qquad |\frac{1}{x}| < 1$$

That is,

$$|x| < 1 \qquad \textit{and} \qquad |x| > 1$$

must both be true for the original statement to be accurate. However, by the axiom of extentionality, a value cannot be simultaneously greater than and less than one.

The two functions have non-coinciding radii of convergence and therefore, for any value of x, at least one of the summations diverges to infinity (both diverge for $x = 1$). Therefore, there is no value of x for which both converge, and hence no condition in which the two series are mutually algebraically workable. Therefore, the initial contention is false.

3 Components of Exponents

The formulas for the sums of the first x integers, taken to select exponents are as follows:

$$\sum_{n=1}^{x} n^0 = x$$
$$\sum_{n=1}^{x} n^1 = \frac{x(x+1)}{2}$$
$$\sum_{n=1}^{x} n^2 = \frac{x(x+1)(2x+1)}{6}$$
$$\sum_{n=1}^{x} n^3 = \frac{x^2(x+1)^2}{4}$$
$$\sum_{n=1}^{x} n^4 = \frac{x(x+1)(2x+1)(3x^2+3x-1)}{30}$$
$$\sum_{n=1}^{x} n^5 = \frac{x^2(x+1)^2(2x^2+2x-1)}{12}$$
$$\sum_{n=1}^{x} n^6 = \frac{x(x+1)(2x+1)(3x^4+6x^3-3x+1)}{42}$$

Each $\sum_{n=1}^{x} n^p$ is evaluable as a polynomial of the order $(p-1)$. As such, it is possible to evaluate x^p as a summation where x is any positive integer and p is any positive exponent by manipulating summations such as these.

For the first few values of p, it can be shown that:

$$x^1 = x = \sum_{n=1}^{x} n^0 = \sum_{n=1}^{x} 1$$

$$\begin{aligned}
\frac{x(x+1)}{2} &= \sum_{n=1}^{x} n^1 \\
x^2 + x &= 2\sum_{n=1}^{x} n^1 \\
x^2 + \sum_{n=1}^{x} n^0 &= 2\sum_{n=1}^{x} n^1 \\
x^2 &= 2\sum_{n=1}^{x} n^1 - \sum_{n=1}^{x} n^0 \\
x^2 &= \sum_{n=1}^{x} [2n-1]
\end{aligned}$$

$$\begin{aligned}
\frac{x(x+1)(2x+1)}{6} &= \sum_{n=1}^{x} n^2 \\
2x^3 + 3x^2 + x &= 6\sum_{n=1}^{x} n^2 \\
2x^3 + 3\sum_{n=1}^{x} [2n-1] + \sum_{n=1}^{x} 1 &= 6\sum_{n=1}^{x} n^2 \\
2x^3 + 6\sum_{n=1}^{x} n - 3\sum_{n=1}^{x} 1 + \sum_{n=1}^{x} 1 &= 6\sum_{n=1}^{x} n^2 \\
2x^3 &= 6\sum_{n=1}^{x} n^2 - 6\sum_{n=1}^{x} n + 2\sum_{n=1}^{x} 1 \\
x^3 &= \sum_{n=1}^{x} \left[3n^2 - 3n + 1\right]
\end{aligned}$$

$$\frac{x^2(x+1)^2}{4} = \sum_{n=1}^{x} n^3$$

$$x^4 + 2x^3 + x^2 = 4\sum_{n=1}^{x} n^3$$

$$x^4 + 2\sum_{n=1}^{x}\left[3n^2 - 3n + 1\right] + \sum_{n=1}^{x}[2n-1] = 4\sum_{n=1}^{x} n^3$$

$$x^4 + 6\sum_{n=1}^{x} n^2 - 6\sum_{n=1}^{x} n + 2\sum_{n=1}^{x} 1 + 2\sum_{n=1}^{x} n - \sum_{n=1}^{x} 1 = 4\sum_{n=1}^{x} n^3$$

$$x^4 = 4\sum_{n=1}^{x} n^3 - 6\sum_{n=1}^{x} n^2 + 4\sum_{n=1}^{x} n - \sum_{n=1}^{x} 1$$

$$x^4 = \sum_{n=1}^{x}\left[4n^3 - 6n^2 + 4n - 1\right]$$

Omitting the lengthy algebra,

$$x^5 = \sum_{n=1}^{x}\left[5n^4 - 10n^3 + 10n^2 - 5n + 1\right]$$

$$x^6 = \sum_{n=1}^{x}\left[6n^5 - 15n^4 + 20n^3 - 15n^2 + 6n - 1\right]$$

$$x^7 = \sum_{n=1}^{x}\left[7n^6 - 21n^5 + 35n^4 - 35n^3 + 21n^2 - 7n + 1\right]$$

The trend appears to be:

$$x^p = \sum_{n=1}^{x}\left[\sum_{k=1}^{p}\left[\binom{p}{k}(-1)^{k+1}n^{p-k}\right]\right]$$

My proof for this is actually relatively simple, and is demonstrated on the next page.

$$Lemma\ 1: \sum_n^m a_n - \sum_n^{m-1} a_n = a_m$$

$$Lemma\ 2: (n-1)^p = \sum_{k=0}^{p} \binom{p}{k} (-1)^k n^{p-k}$$

By Lemma 1, and remembering that $0^p = 0$ for all positive p,

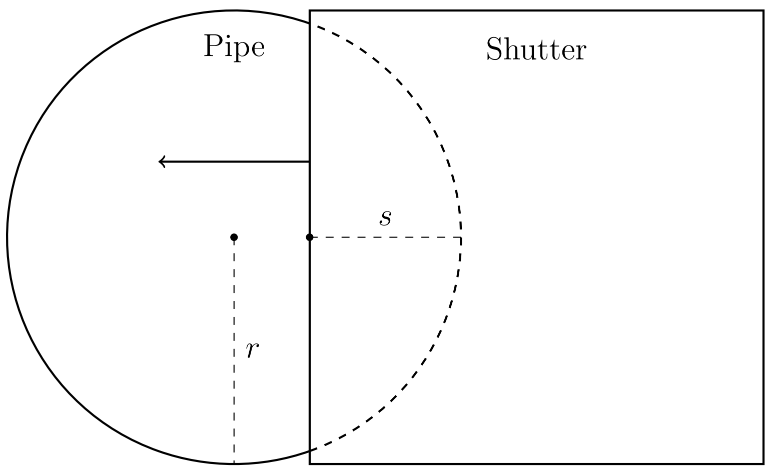
$$\begin{aligned} x^p &= \sum_{n=1}^{x} n^p - \sum_{n=1}^{x-1} n^p \\ &= \sum_{n=1}^{x} n^p - \sum_{n=0}^{x-1} n^p \end{aligned}$$

Then, by shifting the frame of the subtrahend:

$$\begin{aligned} x^p &= \sum_{n=1}^{x} n^p - \sum_{n=1}^{x} (n-1)^p \\ &= \sum_{n=1}^{x} [n^p - (n-1)^p] \\ &= \sum_{n=1}^{x} \left[n^p - \sum_{k=0}^{p} \binom{p}{k} (-1)^k n^{p-k} \right] \\ &= \sum_{n=1}^{x} \left[n^p - \left[n^p + \sum_{k=1}^{p} \binom{p}{k} (-1)^k n^{p-k} \right] \right] \\ &= \sum_{n=1}^{x} \left[(-1) * \sum_{k=1}^{p} \binom{p}{k} (-1)^k n^{p-k} \right] \\ x^p &= \sum_{n=1}^{x} \left[\sum_{k=1}^{p} \left[\binom{p}{k} (-1)^{k+1} n^{p-k} \right] \right] \end{aligned}$$

4 A Challenge in the Realm of Applied Math

I happened upon the following problem: I am given an air pipe of radius r, and told to block the air flow using a square shutter by sliding the shutter over the opening of the pipe, as pictured below. The rate of pipe blockage should be a constant five percent of the total area each minute. Find the function of the rate at which I should slide the shutter.



Shutter moving to cover the pipe opening

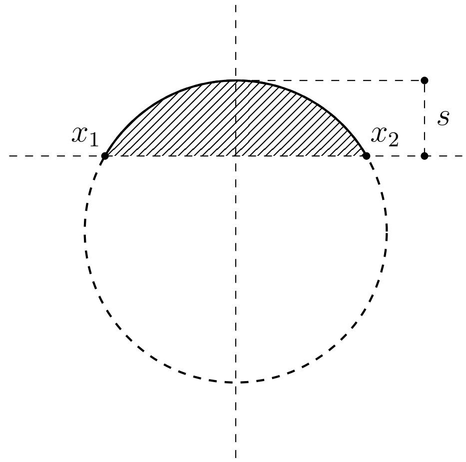
The value s is the horizontal distance that the shutter has covered the pipe. This is essentially a problem of finding the area of a section of a circle, partitioned by a chord (the side of the shutter). Beginning this problem, I had no knowledge of a formula for such an area. So, I set out to derive one.

My first instinct was to model the situation using calculus. The circle could be drawn onto a plane as shown on the following page, turning the problem into a simple area-under-the-curve question. Call the circle function $C(x)$.

$$[C(x) + r - s]^2 + x^2 = r^2$$

or

$$C(x) = \sqrt{r^2 - x^2} - r + s$$



The area of the pipe covered, call it $A(s)$, is the integral of C from x_1 to x_2, as shown, with respect to x.

$$A(s) = \int_{x_1}^{x_2} C(x)\, dx$$

How would one evaluate the limits? x_1 and x_2 are the x-values of the intersection of $C(x)$ and the x-axis. So, they are the x-values that yield $C(x_1) = C(x_2) = 0$.

$$\begin{aligned} 0 &= \sqrt{r^2 - (x_1)^2} - r + s \\ r - s &= \sqrt{r^2 - (x_1)^2} \\ (r-s)^2 &= r^2 - (x_1)^2 \\ r^2 - (r-s)^2 &= (x_1)^2 \\ \pm\sqrt{r^2 - (r-s)^2} &= x_1 \; or \; x_2 \end{aligned}$$

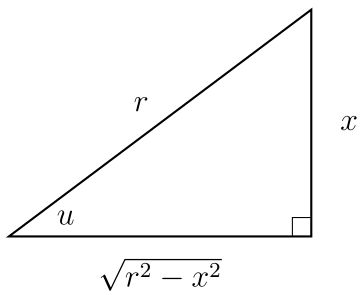
x_1 is the lower bound, and the root itself can never be negative, so it is appropriate to assign each bound as follows:

$$\begin{aligned} x_1 &= -\sqrt{r^2 - (r-s)^2} \\ x_2 &= \sqrt{r^2 - (r-s)^2} \end{aligned}$$

Now comes the difficult part: evaluating the integral.

$$
\begin{aligned}
A(s) &= \int_{x_1}^{x_2} C(x)\, dx \\
&= \int_{x_1}^{x_2} [\sqrt{r^2 - x^2} - r + s]\, dx
\end{aligned}
$$

The best way I know to deal with the root term is by using trigonometric substitution. Define a right triangle like so:



$$sin(u) = \tfrac{x}{r} \qquad cos(u) = \tfrac{\sqrt{r^2-x^2}}{r}$$

$$x = rsin(u) \qquad u = arctan\left(\tfrac{x}{\sqrt{r^2-x^2}}\right)$$

$$dx = rcos(u)\, du$$

$$
\begin{aligned}
A(s) &= \int_{x=x_1}^{x=x_2} [\sqrt{r^2 - (rsin(u))^2} - r + s] * rcos(u)\, du \\
&= \int_{x=x_1}^{x=x_2} [\sqrt{r^2 - r^2 sin^2(u)} - r + s] * rcos(u)\, du \\
&= \int_{x=x_1}^{x=x_2} [\sqrt{r^2(1 - sin^2(u))} - r + s] * rcos(u)\, du \\
&= \int_{x=x_1}^{x=x_2} [\sqrt{r^2(cos^2(u))} - r + s] * rcos(u)\, du \\
&= \int_{x=x_1}^{x=x_2} [rcos(u) - r + s] * rcos(u)\, du \\
&= \int_{x=x_1}^{x=x_2} [r^2 cos^2(u) - r^2 cos(u) + rscos(u)]\, du
\end{aligned}
$$

Keep in mind through all of this that r is a constant.

$$
\begin{aligned}
A(s) &= \int_{x=x_1}^{x=x_2} [r^2cos^2(u) - r^2cos(u) + rscos(u)]\ du \\
&= r^2\left(\int_{x=x_1}^{x=x_2} cos^2(u)\ du\right) - r^2\left(\int_{x=x_1}^{x=x_2} cos(u)\ du\right) + rs\left(\int_{x=x_1}^{x=x_2} cos(u)\ du\right) \\
&= \left[\frac{r^2}{2}(sin(u)cos(u) + u) - r^2sin(u) + rssin(u)\right]_{x=x_1}^{x=x_2}
\end{aligned}
$$

Remember the identities determined from the right triangle on the previous page.

$$
\begin{aligned}
A(s) &= \left[\frac{r^2}{2}\left(\frac{x * \sqrt{r^2 - x^2}}{r * r} + arctan\left(\frac{x}{\sqrt{r^2 - x^2}}\right)\right) - r^2\frac{x}{r} + rs\frac{x}{r}\right]_{x=x_1}^{x=x_2} \\
&= \left[\frac{x\sqrt{r^2 - x^2}}{2} + \frac{r^2}{2}arctan\left(\frac{x}{\sqrt{r^2 - x^2}}\right) - rx + sx\right]_{x=x_1}^{x=x_2} \\
&= \left[x\left(\frac{1}{2}\sqrt{r^2 - x^2} - r + s\right) + \frac{r^2}{2}arctan\left(\frac{x}{\sqrt{r^2 - x^2}}\right)\right]_{x=x_1}^{x=x_2}
\end{aligned}
$$

Now what remains is to plug in the limits.

$$
\begin{aligned}
&= \left[x\left(\frac{1}{2}\sqrt{r^2 - x^2} - r + s\right) + \frac{r^2}{2}arctan\left(\frac{x}{\sqrt{r^2 - x^2}}\right)\right]_{x=-\sqrt{r^2-(r-s)^2}}^{x=\sqrt{r^2-(r-s)^2}} \\
&= \sqrt{r^2 - (r-s)^2}\left(\frac{1}{2}\sqrt{r^2 - \sqrt{r^2 - (r-s)^2}^2} - r + s\right) + \frac{r^2}{2}arctan\left(\frac{\sqrt{r^2 - (r-s)^2}}{\sqrt{r^2 - \sqrt{r^2 - (r-s)^2}^2}}\right) \\
&-(-\sqrt{r^2 - (r-s)^2})\left(\frac{1}{2}\sqrt{r^2 - (-\sqrt{r^2 - (r-s)^2})^2} - r + s\right) \\
&-\frac{r^2}{2}arctan\left(\frac{(-\sqrt{r^2 - (r-s)^2})}{\sqrt{r^2 - (-\sqrt{r^2 - (r-s)^2})^2}}\right)
\end{aligned}
$$

Not exactly eloquently stated, but simplification is possible.

Considering that the *arctan* function is odd, the *arctan* term in the subtrahend is actually just the negative of that in the minuend. In fact, both the third and fourth terms are each the negative of the first and second terms. So, the entire function simplifies to the following.

$$A(s) = 2\sqrt{r^2-(r-s)^2}\left(\frac{1}{2}\sqrt{r^2-\sqrt{r^2-(r-s)^2}^2}-r+s\right)+r^2 arctan\left(\frac{\sqrt{r^2-(r-s)^2}}{\sqrt{r^2-\sqrt{r^2-(r-s)^2}^2}}\right)$$

For further simplification, it is worth noting that the radical components appear multiple times each. Therefore, it is a good idea to simplify each of them independently from the rest of the function.

$$\begin{aligned}
\sqrt{r^2-(r-s)^2} &= \sqrt{r^2-(r^2-2rs+s^2)} \\
&= \sqrt{2rs-s^2} \\
\sqrt{r^2-\sqrt{r^2-(r-s)^2}^2} &= \sqrt{r^2-(r^2-(r-s)^2)} \\
&= \sqrt{(r-s)^2} \\
&= r-s
\end{aligned}$$

So, it is now easier to simplify the original function.

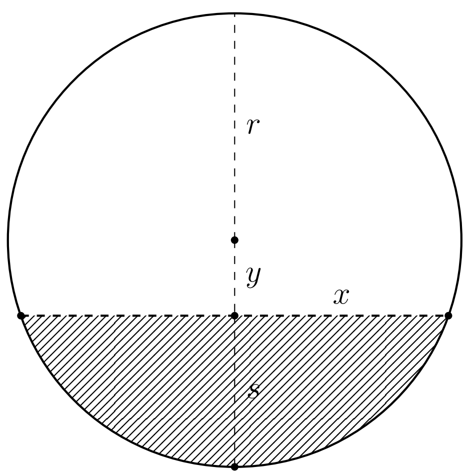
$$\begin{aligned}
A(s) &= 2\sqrt{2rs-s^2}\left(\frac{1}{2}(r-s)-r+s\right)+r^2 arctan\left(\frac{\sqrt{2rs-s^2}}{r-s}\right) \\
&= 2\sqrt{2rs-s^2}\left((\frac{1}{2}-1)(r-s)\right)+r^2 arctan\left(\frac{\sqrt{2rs-s^2}}{r-s}\right) \\
&= 2\left(-\frac{1}{2}(r-s)\right)\sqrt{2rs-s^2}+r^2 arctan\left(\frac{\sqrt{2rs-s^2}}{r-s}\right) \\
&= (s-r)\sqrt{2rs-s^2}+r^2 arctan\left(\frac{\sqrt{2rs-s^2}}{r-s}\right)
\end{aligned}$$

To find the rate to move the shutter, the next step is to take the derivative of both sides with respect to time, then set the $\frac{ds}{dt}$ term to be dependent on the $\frac{dA}{dt}$ term(s). The derivatives are:

$$\frac{dA}{dt} = \frac{ds}{dt}\sqrt{2rs - s^2} + \frac{(s-r)\left[2r\frac{ds}{dt} - 2s\frac{ds}{dt}\right]}{2\sqrt{2rs - s^2}} - \frac{r^2\frac{(r-s)\left[2r\frac{ds}{dt} - 2s\frac{ds}{dt}\right]}{2\sqrt{2rs-s^2}}\sqrt{2rs - s^2}\frac{ds}{dt}}{\left(1 + \frac{2rs-s^2}{r^2-2rs+s^2}\right)(r-s)^2}$$

I have never had the opportunity to take a course in differential equations, so I do not know how to proceed with this strategy for solving the original problem. However, I can think of another way to solve it.

The area covered by the shutter is equal to the distance s, multiplied by the average width of the area of the circle that has been covered. Average width is the integral of the width from $s = 0$ to s. This is modeled below.



$$Width,\ w = 2x \qquad w = 2\sqrt{r^2 - y^2}$$

$$x = \tfrac{w}{2} \qquad y = r - s$$

$$x^2 + y^2 = r^2 \qquad w = 2\sqrt{r^2 - (r-s)^2}$$

$$\left(\tfrac{w}{2}\right)^2 = r^2 - y^2 \qquad w = 2\sqrt{2rs - s^2}$$

$$\begin{aligned} Average\ width,\ W &= 2 * \frac{1}{s-0}\int_0^s \sqrt{2rs - s^2}\ ds \\ &= \frac{2}{s}\int_0^s \sqrt{2rs - s^2}\ ds \end{aligned}$$

It is not necessary to evaluate this integral just yet.

So to reiterate, area covered is the product of the distance the shutter has slid and the average width it has covered. That is,

$$\begin{aligned} A &= W * s \\ &= \left[\frac{2}{s}\int_0^s \sqrt{2rs - s^2}\; ds\right] * s \\ &= 2\int_0^s \sqrt{2rs - s^2}\; ds \end{aligned}$$

So, to find the rate of change of A, we take the derivative with respect to time.

$$\begin{aligned} \frac{d}{dt}A &= \frac{d}{dt}\left[2\int_0^s \sqrt{2rs - s^2}\; ds\right] \\ &= 2\frac{d}{dt}\left[\int_0^s \sqrt{2rs - s^2}\; ds\right] \\ \frac{dA}{dt} &= 2\sqrt{2rs - s^2}\,\frac{ds}{dt} \end{aligned}$$

This final step is of course by the fundamental theorem of calculus. So, we now evaluate the intended value of $\frac{dA}{dt}$, which is 5% of the total area each minute:

$$\begin{aligned} \frac{dA}{dt} &= 5\% * A(2r) \\ &= 0.05 * \pi r^2 \\ &= \frac{\pi r^2}{20} \end{aligned}$$

So,

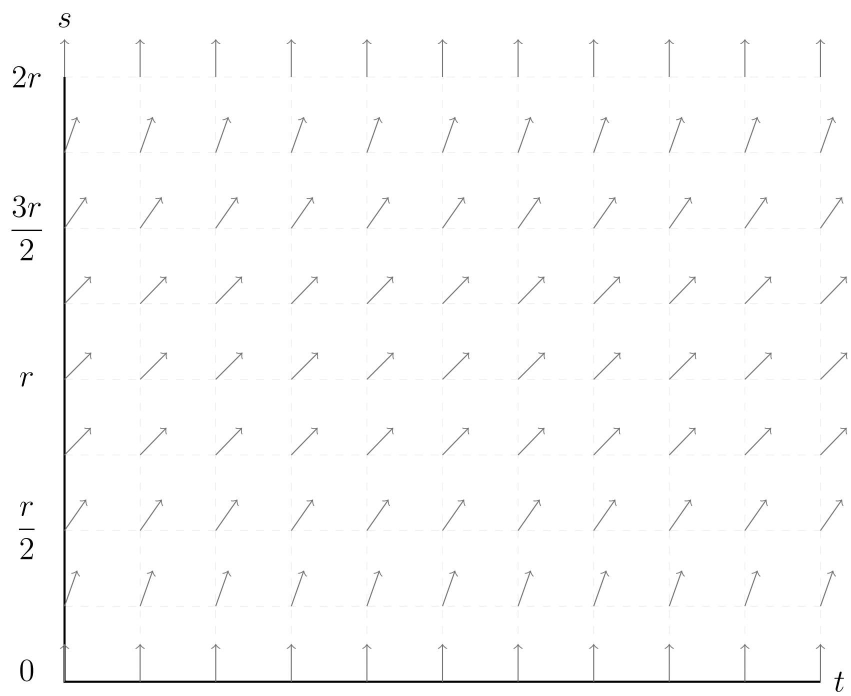
$$\frac{\pi r^2}{20} = 2\sqrt{2rs - s^2}\,\frac{ds}{dt}$$

Therefore, the desired form of the equation is:

$$\frac{ds}{dt} = \frac{\pi r^2}{40\sqrt{2rs - s^2}}$$

Again, I do not have much experience solving differential equations such as this one. But, it is simple enough that I can model the solution by constructing a slope field.

Slope Field for $\frac{ds}{dt} = \frac{\pi r^2}{40\sqrt{2rs-s^2}}$



Observationally, the graph of the antiderivative appears to be akin to the graph of the inverse sine ($arcsin$) function. Another thing to note is that the rate approaches infinity close to $s = 0$ and $s = 2r$. This is logical, as a circle has zero lateral width at its top and bottom. So, the shutter must move with infinite speed to maintain the desired rate of pipe coverage.