The Iterated St. Petersburg Game

In the St. Petersburg game, a player flips a fair coin until she obtains Tails. The player then receives $\$2^N$, where N is the number of times she flipped the coin. The game boasts infinite expected gain for the player, so a person (whose utility function is unbounded and increases linearly as a function of dollars gained) should, apparently, be willing to surrender any finite cost in order to play the game once.

outcome	# of flips	gain	probability
T	1	$2	½
HT	2	$4	¼
HHT	3	$8	⅛
…			

Table 1: Outcome table for the St. Petersburg game

$$\text{EV} = (\$2)\tfrac{1}{2} + (\$4)\tfrac{1}{4} + (\$8)\tfrac{1}{8} + \ldots = \$1 + \$1 + \$1 + \ldots$$

Of course, if a person is given the opportunity to play the game many times – at a finite cost, per game – she should accept: Since each round of the game benefits the player (in expectation), it seems clear that multiple rounds of the game promises a greater benefit. Surprisingly, however, this inference can fail to hold when a person is offered the opportunity to play the game infinitely many times – even when the cost to play is finite in each case. The trick is that the game becomes increasingly expensive.

The Iterated St. Petersburg Game

A person is offered the opportunity to play infinitely many rounds of the St. Petersburg game. Each round, the cost to play almost doubles: The first round costs $3; the second round costs $5; the third round costs $9, and so on. In general, round K costs $\$1 + \$2^K$.

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<tr><th>round</th><th>gain</th><th>cost</th></tr>
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Table 2: Outcome table for the iterated St. Petersburg game described above

Each round, the cost to play is finite and the expected gain is infinite. So each round stands to benefit the player. Somewhat paradoxically, the entire infinite game – offered as a package – will result in an infinite loss for the player with probability 1. Let us see why.

We will say that the player *wins* a round of this infinite game when she gains more from the round than she paid, in that round. And we'll say that she *loses* a round when the cost of the round exceeds the gain by at least a dollar. Note that, as defined, winning and losing exhaust all possibilities. (Strictly speaking, we should add to the rules that if the player *never* flips tails in a given round, there is no award whatsoever for that round.)

With all this in mind, we can say that the entire game is an *abysmal failure* when the player loses every round. The probability of a game's being an abysmal failure is computed as follows:

$$P(abysmal\ failure) = (\tfrac{1}{2})(\tfrac{3}{4})(\tfrac{7}{8})\ldots = \prod_{i=1}^{\infty} \tfrac{2^i-1}{2^i} \approx .288788$$

An abysmal failure will, of course, result in an infinite loss for the player – since the player loses at least a dollar in every round. But this is not the only way for the player to incur an infinite loss. For example, the player could put together some combination of wins and losses during the first five rounds, and then lose the rest. Let us examine the probability that the player never wins after round 5.

$$P(\mathit{lose}\ r6\ \&\ \mathit{lose}\ r7\ \&\ \ldots) = (\tfrac{63}{64})(\tfrac{127}{128})(\tfrac{255}{256})\ldots = \prod_{i=6}^{\infty} \tfrac{2^i-1}{2^i} \approx .969074$$

This probability is fairly high. But suppose we ignore the first twenty rounds, instead of only the first five. We will find that the probability that the player never wins after round 21 is considerably higher still.

$$P(\mathit{lose}\ r21\ \&\ \mathit{lose}\ r22\ \&\ \ldots) = \prod_{i=21}^{\infty} \tfrac{2^i-1}{2^i} \approx .999999$$

These are special cases of what we will call an *abysmal finish*. In general, we will say that a player has an *abysmal finish* if, for some K, the player loses every round that takes place after round K. Note that any game that includes an abysmal finish will also result in an infinite loss for the player.

What, then, is the probability of a game's including an abysmal finish? Using limits and the dominated convergence theorem it can be shown that the probability of there being an abysmal finish is 1.

$$\lim_{K\to\infty} \prod_{i=K}^{\infty} \tfrac{2^i-1}{2^i} = \lim_{K\to\infty} \prod_{i=0}^{\infty} \tfrac{2^{K+i}-1}{2^{K+i}} = \prod_{i=0}^{\infty} \lim_{K\to\infty} (1 - \tfrac{1}{2^{K+i}}) = \prod_{i=0}^{\infty} (1-0) = 1$$

We can conclude that this game will result in an infinite loss for the player with probability 1, despite that each round boasts \$+∞ EV.