

# Bodies invisible from one point

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## Abstract

We show that there exist bodies with mirror surface invisible from a point in the framework of geometrical optics. In particular, we provide an example of a connected three-dimensional body invisible from one point.

**Mathematics subject classifications:** 37D50, 49Q10

**Keywords:** invisible bodies, billiards, shape optimization.

The issue of invisibility attracts a lot of attention nowadays. Various physical and technological ideas aiming at creation of objects invisible for light rays are being widely discussed. A brief review of the major developments is provided in our recent work [4]. For a more entertaining treatment of the subject the reader may refer to a recent article in the BBC Focus Magazine [1]. Most of the recent developments are based on the wave representation of light (e.g. [3, 6, 7]). However, here we study the notion of invisibility from the viewpoint of geometrical optics. In other words, we consider a model where a bounded open set  $B$  with a piecewise smooth boundary in Euclidean space  $\mathbb{R}^d$ ,  $d \geq 2$  represents a physical body with mirror surface, and the billiard in the complement of this domain,  $\mathbb{R}^d \setminus B$ , represents propagation of light outside the body. This work continues the series of results on invisibility obtained in [2, 4, 5].

A semi-infinite broken line  $l \subset \mathbb{R}^d \setminus B$  with the endpoint at  $O \in \mathbb{R}^d \setminus \bar{B}$  is called a billiard trajectory emanating from  $O$ , if the endpoints of its segments (except for  $O$ ) are regular points of the body boundary  $\partial B$  and the outer normal to  $\partial B$  at any such point is the bisector of the angle formed by the segments adjoining the point.

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**Definition 1.** We say that  $B$  is *invisible from a point*  $O \notin \bar{B}$ , if for almost any ray with the vertex at  $O$  there exists a billiard trajectory emanating from  $O$  with a finite number of segments such that the first segment (adjoining  $O$ ) and the last (infinite) segment belong to the ray (see Fig. 1).

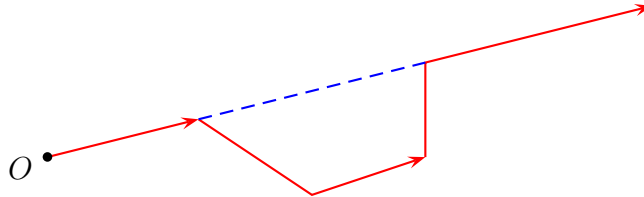


Figure 1: A billiard trajectory emanating from  $O$  with the first and last segments contained in a ray with vertex at  $O$  (the body itself is not shown).

The main result of this note is the following

**Theorem 1.** *For each  $d$  there exists a body  $B \subset \mathbb{R}^d$  invisible from a point. If  $d \geq 3$  then the body is connected.*

Not much is known today about invisibility in the billiard setting. In the limit where the point of reference  $O$  goes to infinity the notion of invisibility from a point is transformed into the notion of invisibility in one direction (see [2], [4] and [5]). There exist two- and three-dimensional bodies invisible in one direction [2] and three-dimensional bodies invisible in two orthogonal directions [4]. It is straightforward to generalize these results to obtain  $d$ -dimensional bodies that are invisible in  $d - 1$  mutually orthogonal directions. On the other hand, there are no bodies invisible in *all* directions (or, equivalently, invisible from *all* points).

The proof of the theorem is based on a direct construction. In the proof we use the following lemma.

**Lemma 1** (A characteristic property of a bisector in a triangle). *Consider a triangle  $ABC$  and a point  $D$  lying on the side  $AC$ . Let  $AB = a_1$ ,  $BC = a_2$ ,  $AD = b_1$ ,  $DC = b_2$ , and  $BD = f$  (see Fig. 2). The segment  $BD$  is the bisector of the angle  $\angle ABC$  if and only if*

$$(a_1 + b_1)(a_2 - b_2) = f^2.$$

*Proof.* Consider the following relations on the values  $a_1$ ,  $a_2$ ,  $b_1$ ,  $b_2$ , and  $f$ :

- (a)  $a_1/a_2 = b_1/b_2$ ;
- (b)  $a_1a_2 - b_1b_2 = f^2$ ;

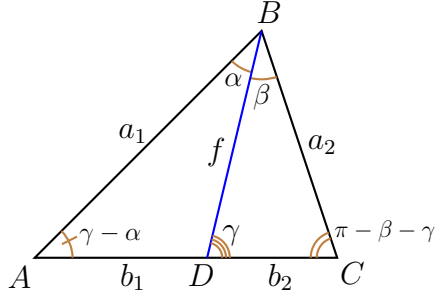


Figure 2: A characteristic property of a bisector in a triangle.

(c)  $(a_1 + b_1)(a_2 - b_2) = f^2$ .

The equalities (a) and (b) are well known; each of them is a characteristic property of triangle bisector as well. It is interesting to note that each of these algebraic relations is a direct consequence of the two others.

Assume that  $BD$  is the bisector of the angle  $\angle ABC$ . Then the equalities (a) and (b) are true, therefore (c) is also true. The direct statement of the lemma is thus proved.

To derive the inverse statement, we need to apply the sine rule and some trigonometry. Denote  $\alpha = \angle ABD$ ,  $\beta = \angle CBD$ , and  $\gamma = \angle BDC$  (see Fig. 2). Applying the sine rule to  $\triangle ABD$ , we have

$$\frac{a_1}{\sin \gamma} = \frac{b_1}{\sin \alpha} = \frac{f}{\sin(\gamma - \alpha)},$$

and applying the sine rule to  $\triangle BDC$ , we have

$$\frac{a_2}{\sin \gamma} = \frac{b_2}{\sin \beta} = \frac{f}{\sin(\gamma + \beta)}.$$

This implies that

$$a_1 + b_1 = \frac{f}{\sin(\gamma - \alpha)} (\sin \gamma + \sin \alpha) = f \frac{\sin \frac{\gamma + \alpha}{2}}{\sin \frac{\gamma - \alpha}{2}},$$

$$a_2 - b_2 = \frac{f}{\sin(\gamma + \beta)} (\sin \gamma - \sin \beta) = f \frac{\sin \frac{\gamma - \beta}{2}}{\sin \frac{\gamma + \beta}{2}}.$$

Using the equality (c), one gets

$$f^2 \frac{\sin \frac{\gamma + \alpha}{2}}{\sin \frac{\gamma - \alpha}{2}} \frac{\sin \frac{\gamma - \beta}{2}}{\sin \frac{\gamma + \beta}{2}} = f^2,$$

whence

$$\sin \frac{\gamma + \alpha}{2} \sin \frac{\gamma - \beta}{2} = \sin \frac{\gamma - \alpha}{2} \sin \frac{\gamma + \beta}{2},$$

After some algebra as a result we have

$$\cos \left( \gamma + \frac{\alpha - \beta}{2} \right) = \cos \left( \gamma - \frac{\alpha - \beta}{2} \right).$$

The last equation and the conditions  $0 < \alpha, \beta, \gamma < \pi$  imply that  $\alpha = \beta$ . The inverse statement of the lemma is also proved.  $\square$

*Proof of Theorem 1.* Consider confocal ellipse and hyperbola on the plane. In a convenient coordinate system in which the major and minor axes of the ellipse coincide with the coordinate axes, the ellipse is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0,$$

and the hyperbola is given by the equation

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1,$$

with the relation

$$c^2 = a^2 - b^2 = \alpha^2 + \beta^2 \tag{1}$$

ensuring that the ellipse and the parabola are confocal. Observe that the two foci  $F_1$  and  $F_2$  are located at  $(\pm c, 0)$ . Finally, we require the intersection of the ellipse with each of the branches of the hyperbola to be in-line with the relevant focus (the dashed line on Fig. 3). It is an elementary exercise to

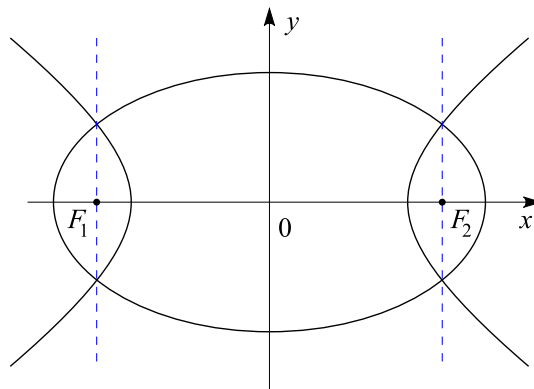


Figure 3: An ellipse and hyperbola satisfying the conditions (1) and (2).

check that this property is guaranteed by the condition

$$\frac{1}{\beta^2} - \frac{1}{b^2} = \frac{1}{c^2}. \quad (2)$$

On Fig. 4 the ellipse is indicated by  $\mathcal{E}$ , the right branch of the hyperbola by  $\mathcal{H}$  (the other branch is not considered), and the foci by  $F_1$  and  $F_2$ .

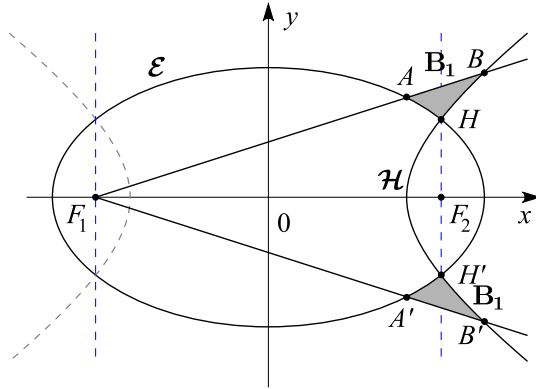


Figure 4: A body having zero resistance to a flow of particles emanating from a point.

Choose an arbitrary point  $B$  on  $\mathcal{H}$ , such that it lies outside of the ellipse, and denote by  $A$  the intersection of the segment  $F_1B$  with the ellipse. Let  $H$  be the closest intersection point with the ellipse as we move from  $B$  along the branch of the hyperbola towards the ellipse (see Fig. 4, where the relevant point  $B$  happens to belong to the top segment of the hyperbola's branch  $\mathcal{H}$ ). We are interested in the curvilinear triangle  $ABH$  and its symmetric (w.r.t. the major axis of the ellipse) counterpart  $A'B'H'$ . We denote the union of the aforementioned curvilinear triangles by  $\mathbf{B}_1$  (shown in grey color on Fig. 4). By construction

$$\angle AF_1F_2 = \angle A'F_1F_2. \quad (3)$$

Consider a particle emanating from  $F_1$  and making a reflection from  $\mathbf{B}_1$ . The first reflection is from one of the arcs  $AH$  or  $A'H'$ . Without loss of generality the point of first reflection  $C$  lies on  $A'H'$  (see Fig. 5). We have

$$\angle CF_1F_2 < \angle A'F_1F_2. \quad (4)$$

After the reflection the particle passes through the focus  $F_2$  and then intersects  $\mathcal{H}$  at a point  $D$  (recall that by construction the segments  $HF_2$  and  $H'F_2$  are orthogonal to the major axis  $F_1F_2$ , hence, the intersection of the ray that

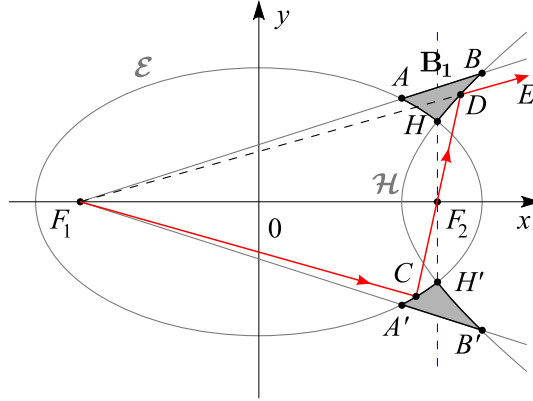


Figure 5: The trajectory of a particle emanating from  $F_1$ .

emerges from  $C$  and passes through  $F_2$  necessarily reaches the hyperbola outside of the ellipse).

By the focal property of ellipse we have

$$|F_1C| + |F_2C| = |F_1H| + |F_2H|, \quad (5)$$

and by the focal property of hyperbola,

$$|F_1D| - |F_2D| = |F_1H| - |F_2H|. \quad (6)$$

Multiplying both parts of (5) and (6) and bearing in mind that  $F_1F_2$  is orthogonal to  $F_2H$ , we obtain

$$(|F_1C| + |F_2C|)(|F_1D| - |F_2D|) = |F_1H|^2 - |F_2H|^2 = |F_1F_2|^2. \quad (7)$$

Applying Lemma 1 to the triangle  $CF_1D$  and using (7) we conclude that  $F_1F_2$  is a bisector of this triangle, that is,

$$\angle CF_1F_2 = \angle DF_1F_2. \quad (8)$$

Using (3), (4), and (7), we obtain that  $\angle DF_1F_2 < \angle AF_1F_2$ , therefore  $D$  lies on the arc  $HB$ . After reflecting at  $D$  the particle moves along the line  $DE$  containing  $F_1$ . This property can be interpreted as  $B_1$  having zero resistance to the flow of particles emanating from  $F_1$ .

Now consider the body  $\mathbf{B}_2$  obtained from  $\mathbf{B}_1$  by dilation with the center at  $F_1$  and such that  $\mathbf{B}_1$  and  $\mathbf{B}_2$  have exactly two points in common (in Fig. 6 the dilation coefficient is greater than 1). A particle emanating from  $F_1$  and reflected from  $\mathbf{B}_1$  at  $C$  and  $D$ , further moves along the line  $DE$  containing  $F_1$ , besides the equality (8) takes place.

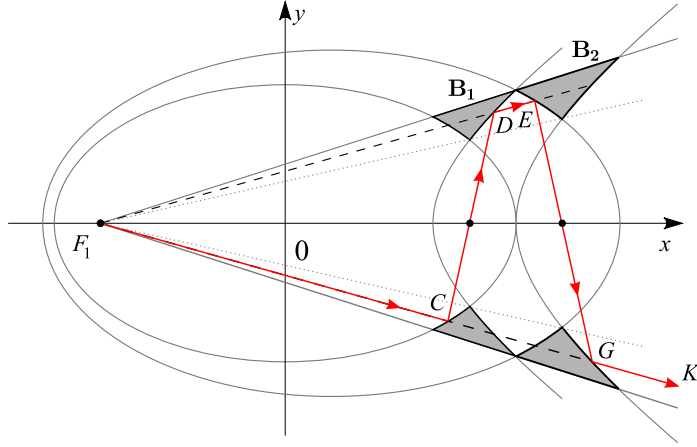


Figure 6: A body invisible from one point.

Then the particle makes two reflections from  $\mathbf{B}_2$  at  $E$  and  $G$  and moves freely afterwards along a line containing  $F_1$ , besides the equality

$$\angle EF_1F_2 = \angle GF_1F_2. \quad (9)$$

takes place. Using (8) and (9), as well as the (trivial) equality  $\angle DF_1F_2 = \angle EF_1F_2$ , we find that

$$\angle CF_1F_2 = \angle GF_1F_2.$$

This means that the initial segment  $F_1C$  of the trajectory and its final ray  $GK$  lie in the same ray  $F_1K$ . The rest of the trajectory, the broken line  $CDEG$ , belongs to the convex hull of the set  $\mathbf{B}_1 \cup \mathbf{B}_2$ . Thus we have proved that  $\mathbf{B}_1 \cup \mathbf{B}_2$  is a two-dimensional body invisible from the point  $F_1$ .

In the case of a higher dimension  $d$  the (connected) body invisible from  $F_1$  is obtained by rotation of  $\mathbf{B}_1 \cup \mathbf{B}_2$  about the axis  $F_1F_2$ : a three-dimensional body is shown on Fig. 7. Observe that because of the rotational symmetry of the body the trajectory of the particle emitted from the relevant focal point (that corresponds to  $F_1$  in the two-dimensional case) lies within a plane that contains the major axis of the relevant ellipsoids.

Another example of a three-dimensional body invisible from a point can be obtained by rotating the two-dimensional construction around the axis perpendicular to the major axes of the ellipses and passing through the focal point  $F_1$  (see Fig. 8). □

**Remark 1.** From the proof of the theorem we see that the invisible body is determined by 5 parameters:  $a$ ,  $b$ ,  $\alpha$ ,  $\beta$ , and the inclination of the line  $F_1B$ , with 2 conditions imposed by (1) and (2). Thus, the construction is defined

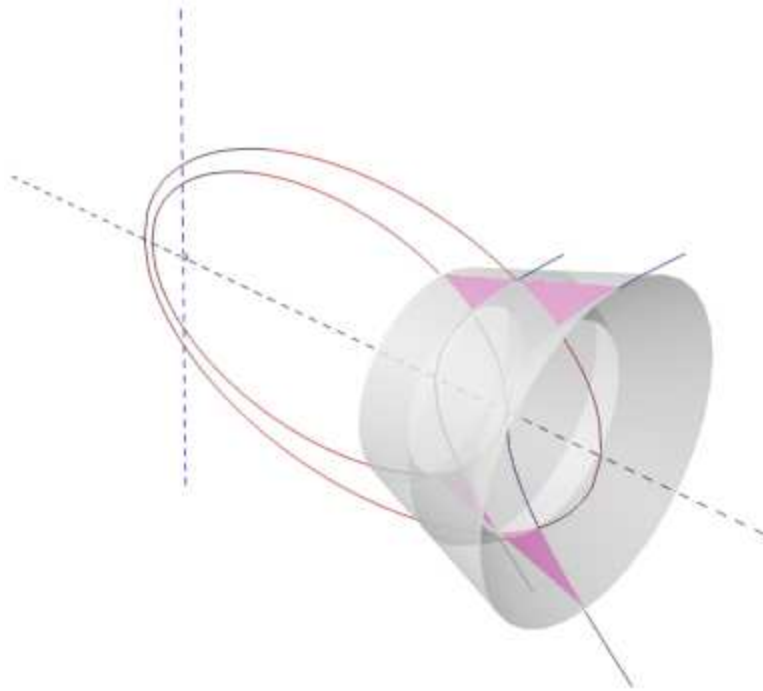


Figure 7: A connected body invisible from one point.

by three parameters. One of them is the scale, and the second and third ones can be taken to be the angles  $\angle HF_1F_2$  and  $\angle BF_1F_2$ .

## Acknowledgements

This work was partly supported by *FEDER* funds through *COMPETE*–Operational Programme Factors of Competitiveness and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (CIDMA) and the Portuguese Foundation for Science and Technology (FCT), within project PEst-C/MAT/UI4106/2011 with *COMPETE* number FCOMP-01-0124-FEDER-022690 and project PTDC/MAT/113470/2009.

## References

- [1] *How to make anything invisible* BBC Focus, Issue 336, December 2011, 33–39.
- [2] A. Aleksenko and A. Plakhov. *Bodies of zero resistance and bodies invisible in one direction*. *Nonlinearity* **22**, 1247-1258 (2009).



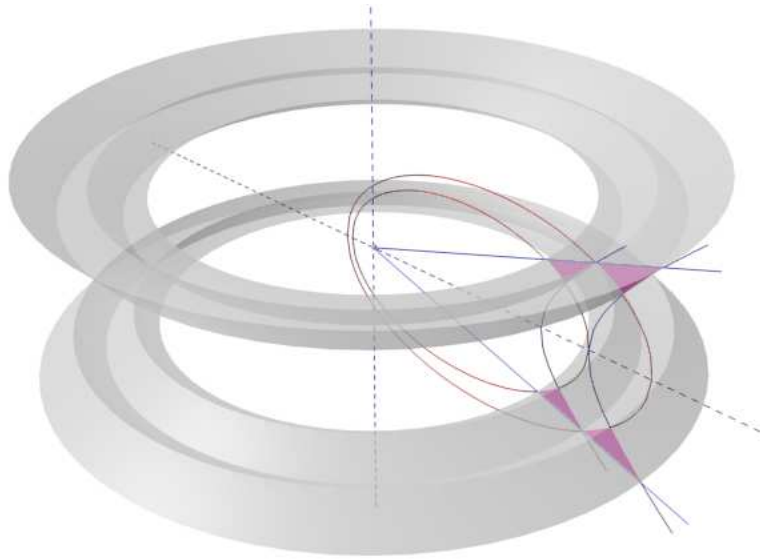


Figure 8: Another body invisible from one point

- [3] T. Ergin, N. Stenger, P. Brenner, J. B. Pendry and M. Wegener. *Three-Dimensional Invisibility Cloak at Optical Wavelengths*. *Science* **328**, 337339 (2010).
- [4] A. Plakhov and V. Roshchina. *Invisibility in billiards*. *Nonlinearity* **24**, 847–854 (2011).
- [5] A. Plakhov and V. Roshchina. *Fractal bodies invisible in 2 and 3 directions*. arXiv:1107.5667 (2011).
- [6] D. Schurig, J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr and D. R. Smith *Metamaterial Electromagnetic Cloak at Microwave Frequencies*, *Science* **314**, 977 (2006).
- [7] J. Valentine, S. Zhang, T. Zentgraf, E. Ulin-Avila, D. A. Genov, G. Bartal and X. Zhang. *Three-dimensional optical metamaterial with a negative refractive index*. *Nature*, 455 (2008).