Spring 2017

MA 262
Study Guide - Exam # 1 (Part A)

1 Solutions to Special Types of 1^{st} Order Equations:

I Separable Equation (SEP):

$$p(y)\,\frac{dy}{dx} = q(x), \text{ or equivalently, } p(y)\,dy = q(x)\,dx$$

Solution Method : Solution $y = y(x)$ is given *implicitly* by

$$\int p(y)\,dy = \int q(x)\,dx$$

⚠ There may be additional constant solutions $y \equiv K$ which arise if you divide by expressions involving y. You must check to see if there are additional solutions.

II First Order Linear Equation (FOL):

$$\frac{dy}{dx} + p(x)\,y = q(x) \quad \text{(Standard Form)}$$

Solution Method : $y = \dfrac{1}{I(x)}\left[\int I(x)\,q(x)\,dx + C\right]$, for an integrating factor $I(x) = e^{\int p(x)\,dx}$.

⚠ The FOL equation **must** be in Standard Form.

III Homogeneous Equation (HOM):

$$\frac{dy}{dx} = f(x, y)\,, \text{ where } f(tx, ty) = f(x, y),\ t > 0$$

Solution Method : Let $V(x) = \dfrac{y}{x}$. Hence $\boldsymbol{y = xV(x)}$ and so $\boldsymbol{\dfrac{dy}{dx} = x\dfrac{dV}{dx} + V}$.

This change of variables transforms the original Homogeneous equation into a Separable equation involving V. Solve the resulting Separable equation and then remember $V = \dfrac{y}{x}$.

⚠ There may be extra solutions.

IV Bernoulli Equation (BER):

$$\frac{dy}{dx} + p(x)y = q(x)y^n\,, \quad (n \neq 0, 1)$$

Solution Method : Divide the equation by y^n to get: $\boldsymbol{y^{-n}\dfrac{dy}{dx} + p(x)y^{1-n} = q(x)}$ $(*)$

Let $u = y^{1-n}$ and so $\dfrac{du}{dx} = (1-n)y^{-n}\,\dfrac{dy}{dx}$. Thus, $y^{-n}\dfrac{dy}{dx} = \dfrac{1}{(1-n)}\,\dfrac{du}{dx}$.

This change of variables transforms the original Bernoulli equation into a First Order Linear equation involving u. Substitute these into $(*)$ and solve the resulting First Order Linear Equation for u and then remember $u = y^{1-n}$.

V **Exact Equation (EXE)**: $M(x,y)\,dx + N(x,y)\,dy = 0, \quad \text{where } \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solution Method : Solution $y = y(x)$ given *implicitly* by an equation $\phi(x,y) = C$ where the function $\phi(x,y)$ is determined by either of these methods:

(a) Book method:

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = M(x,y) & \overset{I_x}{\Longrightarrow} & \phi(x,y) = \displaystyle\int M(x,y)\,dx + h(y) \\ & & \Big\Downarrow D_y \\ \dfrac{\partial \phi}{\partial y} = N(x,y) & = & \dfrac{\partial \phi}{\partial y} = \dfrac{\partial}{\partial y}\left(\displaystyle\int M(x,y)\,dx\right) + h'(y) \quad (\#) \end{cases}$$

$\Longrightarrow$ Find the function $h(y)$ in equation (#) and hence $\phi(x,y) = \int M(x,y)\,dx + h(y)$.

(b) Student method:

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = M(x,y) & \overset{I_x}{\Longrightarrow} & \phi(x,y) = \displaystyle\int M(x,y)\,dx + h(y) \;\; (*) \\ \dfrac{\partial \phi}{\partial y} = N(x,y) & \overset{I_y}{\Longrightarrow} & \phi(x,y) = \displaystyle\int N(x,y)\,dy + g(x) \;\; (**) \end{cases}$$

$\Longrightarrow$ Compare the two forms (*) and (**) and determine suitable $\phi(x,y)$.

The function $\phi(x,y)$ is called a **potential function** for the differential equation.

VI **Other 1^{st} Order Equations**: If possible, use suitable change of variables or other techniques to convert original differential equation to one of the previous types I - V .

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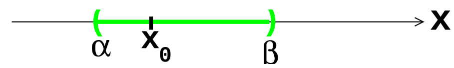
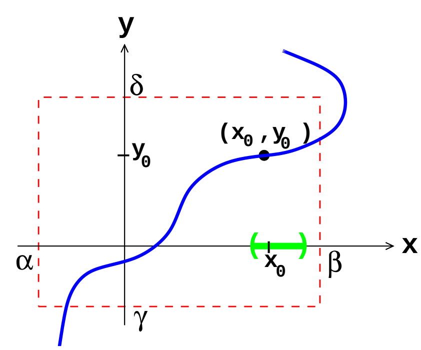
2 Existence and Uniqueness Theorems for 1^{st} Order Equations:

(a) **THEOREM (1^{st} Order Linear)**. If $p(x)$ and $q(x)$ are continuous on an interval I: $\alpha < x < \beta$ containing x_0, then the IVP $\begin{cases} y' + p(x)y = q(x) \\ y(x_0) = y_0 \end{cases}$ has a unique solution $y = \phi(x)$ on the interval I, for any y_0.



Remark: The largest such open interval containing x_0 is where the solution $y = \phi(x)$ is guaranteed to exist. FOL equation must be put in Standard Form.

(b) **THEOREM (1^{st} Order Nonlinear)**. If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are both continuous in some rectangle **R**: $\alpha < x < \beta$, and $\gamma < y < \delta$ and (x_0, y_0) lies inside the rectangle **R**, then the IVP $\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$ has a unique solution $y = \phi(x)$ on *some* interval containing x_0.

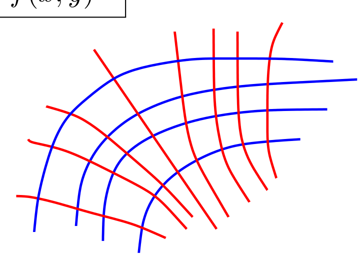


Remarks:

- If the point (x_0, y_0) lies on the boundary of **R**, then this theorem says nothing. The IVP could have no solution, a finite number of solutions, a unique solution, or infinitely many solutions i.e., anything is possible.
- The interval containing x_0 where solution exists can be estimated by looking at the slope/direction field of the differential equation. To determine the exact interval, you must solve the IVP explicitly for y.

3 Applications of 1st Order Equations:

A1 Orthogonal Families of Curves: If a family of given curves $F(x, y, C) = 0$, has slope at (x, y) given by $\dfrac{dy}{dx} = f(x, y)$, then the slope of the tangent line to the family of curves that are *perpendicular/orthogonal* to this family is thus $\boxed{\dfrac{dy}{dx} = -\dfrac{1}{f(x, y)}}$



A2 <u>Malthusian Population Growth</u>: $P(t) =$ population at time t

$$\frac{dP}{dt} = k\,P$$

The solution is $P(t) = P_0 e^{kt}$.

Time to double population, *doubling time*, $t_D = \frac{1}{k}\ln 2$.

A3 <u>Logistic Population Growth</u>: $P(t) =$ population at time t

$$\frac{dP}{dt} = r\,P\left(1 - \frac{P}{C}\right) \quad \text{where} \quad r, C > 0$$

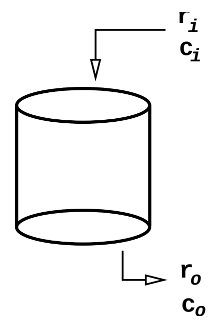
The solution is $P(t) = \dfrac{P_0 C}{P_0 + (C - P_0)\,e^{-rt}}$.

C is called the *Carrying Capacity* of the population.

A4 <u>Mixing Problems</u>: $A(t) =$ amount of substance in tank at time t

$V(t) =$ volume of solution in tank at time t

$$\frac{dA}{dt} = r_i\,c_i - r_o\,c_o, \quad \text{where} \quad c_0 = \frac{A(t)}{V(t)}$$



A5 <u>Newton's Law of Cooling</u>: $T(t) =$ temperature at time t

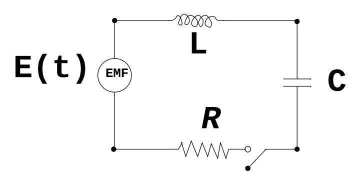
$T_m =$ temperature of surrounding medium

$$\frac{dT}{dt} = -k\,(T - T_m), \; k > 0$$

A6 <u>RLC Circuits</u>: $q(t) =$ charge; $i(t) = \frac{dq}{dt} =$ current

Kirchoff's 2^{nd} *Law* : Voltage drop around closed circuit is zero. Hence

$$L\,\frac{di}{dt} + R\,i + \frac{q}{C} = E(t)$$



Equivalently, $L\,\dfrac{d^2q}{dt^2} + R\,\dfrac{dq}{dt} + \dfrac{q}{C} = E(t)$.

A7 <u>Falling/Rising Objects</u>: Newton's 2^{nd} Law: $F = m\,\dfrac{dv}{dt} = m\dfrac{d^2y}{dt^2}$. Near the surface of the Earth, the force due to gravity is the weight of the object $F_g = mg$. For the two cases below (no air resistance) we have:

$$\text{(a)} \quad m\frac{d^2y}{dt^2} = mg \qquad \text{and} \qquad \text{(b)} \quad m\frac{d^2y}{dt^2} = -mg$$

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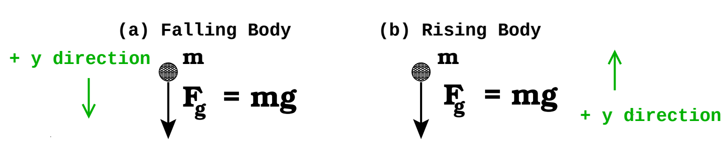


4 **Special Types of 2^{nd} Order Equations** : $\dfrac{d^2y}{dx^2} = F\left(x, y, \dfrac{dy}{dx}\right)$

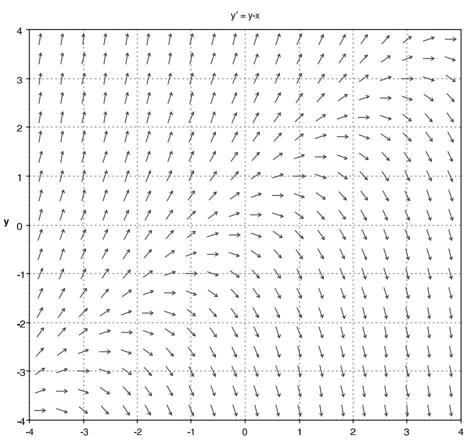
i No Dependent Variable (*Missing* y) : $\dfrac{d^2y}{dx^2} = F\left(x, \dfrac{dy}{dx}\right)$ $(*)$

Use the substitution $\boxed{\dfrac{dy}{dx} = v(x)}$ and hence $\boxed{\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}}$ to convert the 2^{nd} order equation $(*)$ to a 1^{st} order equation in $v(x)$ and solve it for $v(x)$. Then since $\dfrac{dy}{dx} = v(x)$, solve this 1^{st} order equation for y.

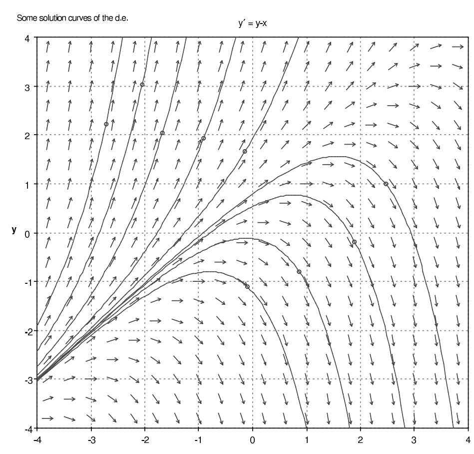
ii No Independent Variable (*Missing* x) : $\dfrac{d^2y}{dx^2} = F\left(y, \dfrac{dy}{dx}\right)$ $(**)$

Use the substitution $\boxed{\dfrac{dy}{dx} = v(y)}$ and, by the Chain Rule, $\boxed{\dfrac{d^2y}{dx^2} = v\,\dfrac{dv}{dy}}$ to convert the 2^{nd} order equation $(**)$ to a 1^{st} order equation in $v(y)$ and solve it for $v(y)$. Then since $\dfrac{dy}{dx} = v(y)$, solve this 1^{st} order equation for y.

5 **Slope Fields (Direction Fields)**: A solution $y = \phi(x)$ to the d.e. $\dfrac{dy}{dx} = f(x, y)$ has slope $f(x, y)$ at the point (x, y). The *slope field* (or *direction field*) of the d.e. indicates the slope of solutions at various points (x, y):



Slope field for $\dfrac{dy}{dx} = f(x, y)$



Slope field and solutions to $\dfrac{dy}{dx} = f(x, y)$

- To sketch slope fields, usually consider where the slopes are constant k. Thus on the curves given by $f(x, y) = k$, the solutions to $\dfrac{dy}{dx} = f(x, y)$ all have constant slope k along these curves. Such curves are called isoclines.
- If the d.e. has a constant solution $y \equiv L$, it is called an equilibrium solution to the d.e.
- The concavity of solutions are determined by the sign of $\dfrac{d^2y}{dx^2}$.
- The direction field may be used to give qualitative information about the behavior of solutions as $x \to \infty$ (or $x \to -\infty$, or $x \to 0$, etc). Slope fields may also be used to estimate the interval where a solution through a point (x_0, y_0) is defined.