## Unit 10 Lesson 6: Surface Integrals

In this lesson we will discuss the concept of a flux integral as well as how to evaluate surface integrals and determine the orientation of a surface.

To introduce the surface integral, view the following video, courtesy of Khan Academy. The video details the development of the integral to help you gain an understanding of where it comes from. While the notation differs from that which we will use, the video provides some intuition into the integral's construct.

**Internet Activity I**

Visit the following website to view the video provided on the Introduction to the Surface Integral. The video is approximately 22 minutes long.

**http://www.khanacademy.org/v/introduction-to-the-surface-integral?p=Calculus**

**Notation:** Note that the video used *A* to represent the region in the plane over which we integrate. We will use *R*. In addition, the video represented the surface with $\Sigma$, while we will use *S*.

The video essentially leaves us with the surface integral in the form of (using our notation),

$$\iint_S f(x,y,z)\,dS$$

Recall from our lesson on surface area (in Unit 10) that the area of a surface $g(x,y)$ over a region *R* in the *xy*-plane is given by

$$dS = \sqrt{1+\left(g_x(x,y)\right)^2+\left(g_y(x,y)\right)^2}\,dA.$$

This leads us to the following theorem.

**Evaluating a Surface Integral**

If $S$ is a surface with equation $z = g(x, y)$, $R$ is its projection onto the $xy$-plane, $g$, $g_x$, and $g_y$ are continuous on $R$, and $f$ is continuous on $S$, then the surface integral of $f$ over $S$ is

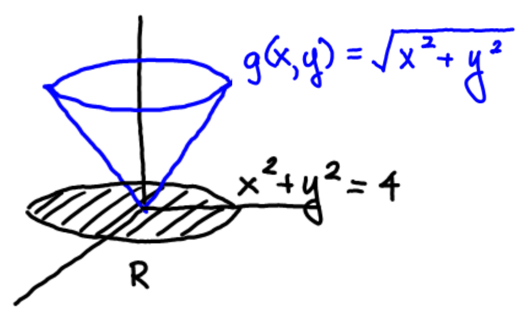
$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{1+\left(g_x(x,y)\right)^2+\left(g_y(x,y)\right)^2}\,dA$$

Let us put this theorem to use to evaluate the surface integral in the next example.

**Example 1:** Evaluate the surface integral $\iint_S f(x,y,z)\,dS$, where $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ and the surface is defined by

$$z = g(x,y) = \sqrt{x^2+y^2}, \text{ and } x^2+y^2 \le 4.$$

**Solution:** Let us first sketch a picture of the surface we are considering. A sketch is provided below:




We calculate the partial derivatives of $z$ as follows:

$$g_x = \frac{x}{\sqrt{x^2+y^2}} \qquad g_y = \frac{y}{\sqrt{x^2+y^2}}$$

So, we have

$$\sqrt{1+\left(g_x(x,y)\right)^2+\left(g_y(x,y)\right)^2} = \sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}$$

$$= \sqrt{1+\left(\frac{x^2}{x^2+y^2}\right)+\left(\frac{y^2}{x^2+y^2}\right)} = \sqrt{\frac{x^2+y^2}{x^2+y^2}+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{2}$$

Now, since $f(x,y,z) = f\left(x,y,\sqrt{x^2+y^2}\right) = \sqrt{x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2} = \sqrt{2\left(x^2+y^2\right)}$, we have the following:

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_R \sqrt{2\left(x^2+y^2\right)}\sqrt{2}\,dA = 2\iint\limits_R \sqrt{x^2+y^2}\,dA$$

In looking at our region *R* and the integrand above, converting this integral to polar coordinates seems appropriate. We do so below and evaluate:

$$2\iint\limits_R \sqrt{x^2+y^2}\,dA = 2\int_0^{2\pi}\int_0^2 r\cdot r\,dr\,d\theta = 2\int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta = 2\int_0^{2\pi}\left[\frac{r^3}{3}\right]_0^2 d\theta = 2\int_0^{2\pi}\frac{8}{3}\,d\theta = \boxed{\frac{32\pi}{3}}$$

**Internet Activity II**

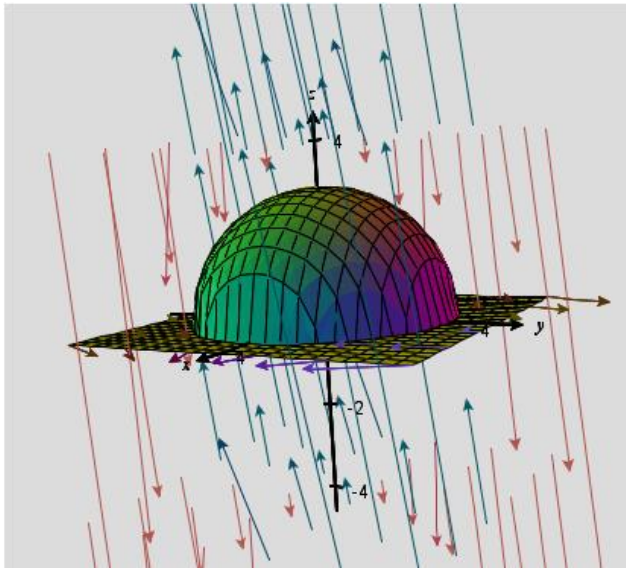
The following two videos provide step-by-step solutions to some examples of surface integrals. Each video is approximately 6-7 minutes in length.

**http://www.youtube.com/watch?v=AUkw5xiVN2U**

**http://www.youtube.com/watch?v=XntA5hj_HBg**

View each one, and follow along on your own paper.

We have just discussed surface integrals where the surface is given explicitly. Here, we begin our study of the surface integral of a vector field over a surface, which are also known as ***flux integrals***. This flux integral ***measures the flow of the vector field across the surface S.*** The figure below illustrates the flow of a vector field through a surface.



Before we begin, we need to establish the concept of an **orientable surface**.

**Definition:** A surface, *S*, is **orientable** if a unit normal vector $\hat{N}$ can be defined at every non-boundary point of *S* in such a way that the normal vectors vary continuously over the surface.

Examples of orientable surfaces are spheres, paraboloids, ellipses and planes.

For orientable surfaces, the gradient vector allows us to easily To calculate flux integrals we will need to find a unit normal vector to a given surface. Consider the orientable surface *S* given by $z = g(x, y)$, and define $G(x, y, z) = z - g(x, y)$. We can think of $g(x, y)$ as a level surface for $G(x, y, z)$, and we know from previous study that the gradient of *G* is perpendicular to any level surface of *G*. Thus the gradient of *G* is perpendicular to the level surface $z = g(x, y)$. *S* can be oriented by the unit normal vector

$$\hat{N} = \frac{\overrightarrow{\nabla G}}{\left\|\overrightarrow{\nabla G}\right\|} = \frac{\left\langle G_x, G_y, G_z \right\rangle}{\left\|\overrightarrow{\nabla G}\right\|} = \frac{\left\langle -g_x, -g_y, 1 \right\rangle}{\sqrt{1 + g_x^{\,2} + g_y^{\,2}}}$$ **upward normal**

OR

$$\hat{N} = \frac{\overrightarrow{\nabla G}}{\left\|\overrightarrow{\nabla G}\right\|} = \frac{\left\langle g_x, g_y, -1 \right\rangle}{\sqrt{1 + g_x^{\,2} + g_y^{\,2}}}$$ **downward normal**

We recall that $\Delta S = \sqrt{1+g_x{}^2+g_y{}^2}\,\Delta A$. So, we have the following flux integral:

$$\iint_S \overrightarrow{F}\cdot\hat{N}\,dS = \iint_R \overrightarrow{F}\cdot\frac{\langle -g_x,-g_y,1\rangle}{\sqrt{1+g_x{}^2+g_y{}^2}}\cdot\sqrt{1+g_x{}^2+g_y{}^2}\,dA$$

$$= \boxed{\iint_R \overrightarrow{F}\langle -g_x,-g_y,1\rangle\,dA} \qquad \text{(oriented upward)}$$

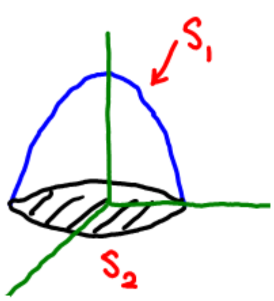
OR $\boxed{\iint_R \overrightarrow{F}\langle g_x,g_y,-1\rangle\,dA}$ (oriented downward)

Let us work together through an example.

**Example 2:** Consider the vector field $\overrightarrow{F}(x,y,z)=(x+y)\hat{i}+y\hat{j}+z\hat{k}$ and the surface enclosed by *S*: $z=1-x^2-y^2$ and $z=0$. Find $\oint\!\!\oint_S \overrightarrow{F}\cdot\hat{N}\,dS$.

**Solution:** We first sketch a picture to vizualize our surface. We consider the surface in two parts, as depicted below, and calculate the flux integral in two parts:

$$\iint_{S_1}\overrightarrow{F}\cdot\overrightarrow{\nabla G}\,dA+\iint_{S_2}\overrightarrow{F}\cdot\overrightarrow{\nabla G}\,dA$$




We first consider the flux across the bottom surface, $S_2$. Here was choose the ***downward normal***, as that would give us the outward flow.

**$S_2$:** $z=g(x,y)=0$ and $\overrightarrow{F}=\langle x+y,y,0\rangle$. So, we have

$$\iint_{S_2}\overrightarrow{F}\langle g_x,g_y,-1\rangle dA=\iint_{S_2}\langle x+y,y,0\rangle\cdot\langle 0,0,-1\rangle dA=\iint_{S_2}0\,dA=\boxed{0}$$

Now, let us consider the flux across the top surface, $S_1$. Here we consider the ***upward normal,*** as that would again give us the outward flow. So, we use $\overrightarrow{\nabla G}=\langle -g_x,-g_y,1\rangle$.

**$S_1$:** $z=g(x,y)=1-x^2-y^2$ and $\overrightarrow{F}=\langle x+y,\ y,1-x^2-y^2\rangle$.

$$\overrightarrow{\nabla G} = \left\langle -g_x, -g_y, 1 \right\rangle = \left\langle 2x,\ 2y, 1 \right\rangle$$

So, $\vec{F} \cdot \overrightarrow{\nabla G} = x^2 + 2xy + y^2 + 1$, and we have the following integral:

$$\iint_{S_1} \vec{F} \cdot \overrightarrow{\nabla G} dA = \iint_{S_1} \left( x^2 + 2xy + y^2 + 1 \right) dA = \int_0^{2\pi} \int_0^1 \left( r^2 + 2r^2 \cos\theta \sin\theta + 1 \right) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left( r^3 + 2r^3 \cos\theta \sin\theta + r \right) dr d\theta = \int_0^{2\pi} \left( \frac{r^4}{4} + \frac{2r^3}{3} \cos\theta \sin\theta + \frac{r^2}{2} \right) \Bigg|_0^1 d\theta$$

$$= \int_0^{2\pi} \left( \frac{1}{4} + \frac{2}{3} \cos\theta \sin\theta + \frac{1}{2} \right) d\theta = \left( \frac{1}{4}\theta + \frac{1}{3}\sin^2\theta + \frac{1}{2}\theta \right) \Bigg|_0^{2\pi} = \frac{2\pi}{4} + \frac{2\pi}{2} = \boxed{\frac{3\pi}{2}}$$

Therefore, our surface integral is equal to $\boxed{\frac{3\pi}{2}}$.

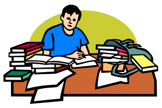
**Internet Activity III**

The following two videos provide step-by-step solutions to some examples of surface integrals of a vector field over a surface. Each video is approximately 6-7 minutes in length.

**http://www.youtube.com/watch?v=y-gsqWf3Gms**

**http://www.youtube.com/watch?v=9H4Q-FEKwGw**

View each one, and follow along on your own paper.

 **Reading Activity**: Read Sec. 15.6 (p. 1112-1114, and p. 1117-1118 only) in the textbook. *Note the box on p. 1121 providing you with a summary of line and surface integrals that may help you organize them for yourself.*

**What To Do Next…** Complete the assignment for Lesson 10-6.