Theorem. *There exists an everywhere-continuous, nowhere-differentiable function.*

Let $h(x) = |x - \lfloor x + 1/2 \rfloor|$, and $f_n(x) = \frac{h(4^n x)}{4^n}$. Then $f_n(x)$ is continuous for every integer $n \geq 0$. We have

$$0 \leq f_n(x) \leq \frac{1}{2 \cdot 4^n} < \frac{1}{4^n}$$

.

Setting $F_m(x) = \sum_{n=0}^{m} f_n(x)$ for integral $m \geq 0$ we have

$$0 \leq F_m(x) \leq F_{m+1}(x)$$

Since

$$F_m(x) < \sum_{n=0}^{m} \frac{1}{4^n} = \frac{1 - \frac{1}{4^{m+1}}}{1 - 1/4} < \frac{4}{3}$$

, for any fixed x there exists

$$\lim_{m \to \infty} F_m(x) = f(x)$$

We will prove first that f is continuous for every x.

For integral m, k with $m > k \geq 0$ and every x, we have

$$0 \leq F_m(x) - F_n(x) = \sum_{n=k+1}^{m} f_n(x) < \sum_{n=k+1}^{m} \frac{1}{4^n} < \frac{1}{4^k}$$

Let $\epsilon > 0$ be given. We choose an integer $k \geq 0$ such that

$$\frac{1}{4^k} < \frac{\epsilon}{3}$$

Then for every x and for integral $m > k$,

$$0 \leq F_m(x) - F_k(x) < \frac{\epsilon}{3}$$

so that

$$0 \leq f(x) - F_k(x) \leq \frac{\epsilon}{3}$$

Hence for every ξ and every h, we have

$$|f(\xi + h) - F_k(\xi + h)| \leq \frac{\epsilon}{3}$$

and

$$|f(\xi) - F_k(\xi)| \leq \frac{\epsilon}{3}$$

Now let ξ be fixed. $F_k(x)$ is continuous at ξ; thus, for suitable $\delta > 0$, for $|h| < \delta$ we have

$$|F_k(\xi + h) - F_k(\xi)| < \frac{\epsilon}{3}$$

so that

$$|f(\xi + h) - f(\xi) < |f(xi + h) - F_k(\xi + h)| + |f(\xi) - F_k(\xi)| + |F_k(\xi + h) - F_k(\xi)| < \epsilon$$

.

Finally we shew that $f(x)$ is not differentiable for any x.

If we had $f'(\xi) = t$ for some value ξ, then for suitable $\delta > 0$ we would have $\left| \frac{f(x) - f(\xi)}{x - \xi} - t \right| < \frac{1}{2}$ for $0 < |x - \xi| < \delta$.

For every sequence ξ_k, for which $\xi_k \neq \xi$, $\xi_k \to \xi$, there would exist a k_0 such that for $k \geq k_0$ we would have

$$\left| \frac{f(\xi_k) - f(\xi)}{\xi_k - xi} - t \right| < \frac{1}{2}$$

and therefore also

$$\left| \frac{f(\xi_{k+1}) - f(\xi)}{\xi_{k+1} - xi} - t \right| < \frac{1}{2}$$

so that

$$\left| \frac{f(\xi_k) - f(\xi)}{\xi_k - xi} - \frac{f(\xi_{k+1}) - f(\xi)}{\xi_{k+1} - \xi} \right| < 1$$

Thus to obtain a contradiction, it suffices to produce a sequence ξ_k, $k \geq 1$ and integral with $\xi_k \neq \xi$, $\xi_k \to \xi$ such that

$$\frac{f(\xi_k) - f(\xi)}{\xi_k - \xi}$$

is an integer for all k and in fact is even if k is even and odd if k is odd.

And this can and will be done..

For integral $k \geq 1$ we set

$$\xi_k =$$

Evidently, we have $\xi_k \neq \xi$, $|\xi_k - \xi| = 4^{-k} \to 0$, $\xi_k \to \xi$.

If n is an integer $\geq k$ then

$$4^n \xi_k = 4^n \xi \pm 4^{n-k}$$

so that $f_n(\xi_k) = f_n(\xi)$. Thus, for integers m, k with $m \geq k \geq 1$,

$$F_m(\xi_k) - F_m(\xi) = \sum_{n=0}^{m} (f_n(\xi_k) - f_n(\xi)) = \sum_{n=0}^{k-1} (f_n(\xi_k) - f_n(\xi))$$

so that for integers $k \geq 1$

$$f(\xi_k) - f(\xi) = \sum_{n=0}^{k-1} (f_n(\xi) - f_n(\xi))$$

If n is an integer such that $0 \leq n \leq k-1$ then

$$2 \cdot 2^{-k} = 2^{-2n-1}$$

Thus

$$4^{-k} \leq 2^{-2n-1} - 2^{-k}$$

Setting

$$a = \lfloor 2^{2n+1} \xi \rfloor$$

we have

$$a \leq 2^{2n+1} \xi < a + 1$$

$$2^{-2n-1} a \leq \xi < 2^{-2n-1}(a+1)$$

I assert that we have

$$2^{-2n-1}a \leq \xi_k < 2^{-2n-1}(a+1)$$

If a is even, then it follows that $f_n(\xi_k) - f_n(\xi) = \xi_k - \xi$, and if a is odd, it follows that $f_n(\xi_k) - f_n(\xi) = -(\xi_k - \xi)$.

In either case $\frac{f_n(\xi_k)-f_n(\xi)}{\xi_k-\xi} = \pm 1$.

Tus if $k \geq 1$ we have

$$\frac{f_n(\xi_k) - f_n(\xi)}{\xi_k - \xi} = \sum_{n=0}^{k-1} \pm 1$$

and so is even for even k, and odd for odd k.